PART A

Unit - III

Complex Variables - 1

3.1 Introduction

We are well acquainted with several concepts associated with a real valued function $y = f(x)$. We introduce *complex valued function* $w = f(z)$ [*function of a complex variable* z] and discuss some topics associated with it.

3.2 Recapitulation of Basic Concepts

A number of the form $z = x + iy$ where x, y are real numbers and $i = \sqrt{-1}$ or $i^2 = -1$ is called a *complex number*. x is called the *real part* of z and y is called the *imaginary part* of z.

Also $\bar{z} = x - iy$ is called the *complex conjugate* of z.

We have $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$\therefore$ $$e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots$$

ie., $$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

or $$e^{ix} = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

Thus $$e^{ix} = \cos x + i \sin x \qquad \ldots (1)$$

by Maclaurin's series.

Hence $e^{i(-x)} = \cos(-x) + i\sin(-x)$

or $$e^{-ix} = \cos x - i \sin x \qquad \ldots (2)$$

Adding and subtracting (1) with (2) we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$\therefore \quad \cos(ix) = \dfrac{e^{i(ix)} + e^{-i(ix)}}{2} = \dfrac{e^{-x}+e^{x}}{2} = \cosh x$

$\sin(ix) = \dfrac{e^{i(ix)} - e^{-i(ix)}}{2i} = \dfrac{e^{-x}-e^{x}}{2i} = \dfrac{i(e^{-x}-e^{x})}{2i^2} = \dfrac{i(e^{x}-e^{-x})}{2} = i \sin hx$

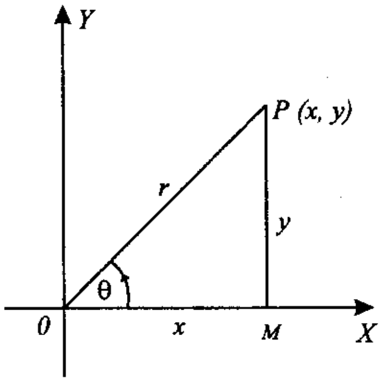
Thus $\cos(ix) = \cos hx$ and $\sin(ix) = i \sin hx$

De-Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$, *where n is a real number.*

Geometrical representation of $z = x + iy$ (*Argand diagram*)

We plot the point $P(x, y)$ in the $x - y$ plane and draw PM perpendicular to the X-axis.



Let $OP = r$ and $\hat{POM} = \theta$. From the figure we have

$\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$

$\therefore \quad x = r\cos\theta,\ y = r\sin\theta$

Eliminating θ (*by squaring and adding*) and r (*by dividing*) we obtain,

$x^2 + y^2 = r^2$ and $y/x = \tan\theta$

or $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$

Since $\cos\theta + i\sin\theta = e^{i\theta}$, $z = r\,e^{i\theta}$ is called the *polar form* of z.

$r = \sqrt{x^2+y^2}$ is called the *modulus* of z and $\theta = \tan^{-1}(y/x)$ is called the *amplitude* of z or *argument* of z. Symbolically we write these as follows. $|z| = r = \sqrt{x^2+y^2}$ and *amp* z or *arg* z $= \theta = \tan^{-1}(y/x)$

Properties associated with the modulus and amplitude

1. (a) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

   (b) $\text{amp}\,(z_1 \cdot z_2) = \text{amp}\, z_1 + \text{amp}\, z_2$

2. (a) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

   (b) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\, z_1 - \text{amp}\, z_2$

3. $|z_1 + z_2| \le |z_1| + |z_2|$

4. $|z_1 - z_2| \ge |z_1| - |z_2|$

Neighbourhood : A neighbourhood of a point z_0 in the complex plane is the set of all points z such that $|z - z_0| < \delta$ where δ is a small positive real number.

Geometrical meaning : If $z_0 = x_0 + i\,y_0$ then

$$|z - z_0| = |(x+iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)|$$

ie., $$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Now $|z - z_0| = \delta$ is $\sqrt{(x - x_0)^2 + (y - y_0)^2} = \delta$

i.e., $$(x - x_0)^2 + (y - y_0)^2 = \delta^2$$

This represents a circle with centre (x_0, y_0) and radius δ.

Geometrically a neighbourhood of a point z_0 (*i.e.,* $|z - z_0| < \delta$) is the set of all points inside a circle having z_0 as the centre and δ as the radius.

3.3 Function of a Complex Variable, Limit, Continuity and Differentiability

Function of a complex variable

If it is possible to find one or more complex numbers w for every value z in a certain domain D, we say that w is a function of z defined for the domain D. In otherwords $w = f(z)$ is called a *function of the complex variable z*. w is said to be single valued or many valued function of z according as for a given value of z there corrresponds one or more than one value of w.

Since $z = x + iy$ or $z = re^{i\theta}$ we always write

$$w = f(z) = u(x, y) + iv(x, y) \quad [\textit{Cartesian form}]$$

$$w = f(z) = u(r, \theta) + iv(r, \theta) \quad [\textit{Polar form}]$$

Examples :

1. Consider $f(z) = z^2$

i.e., $\quad u + iv = (x + iy)^2 = x^2 + 2xiy + i^2 y^2$

or $\quad u + iv = (x^2 - y^2) + i(2xy)$

$\Rightarrow \quad u = x^2 - y^2$ and $v = 2xy$ in the cartesian form.

Also in the polar form $f(z) = f(re^{i\theta}) = (re^{i\theta})^2$

i.e., $\quad u + iv = r^2 e^{2i\theta}$

$$= r^2(\cos 2\theta + i \sin 2\theta)$$

$\Rightarrow \quad u = r^2 \cos 2\theta$ and $v = r^2 \sin 2\theta$

2. Consider $f(z) = \log z$

It is convenient to find u and v in the polar form by taking $z = re^{i\theta}$

We have, $u + iv = \log(re^{i\theta}) = \log r + i\theta \log e$. But $\log_e e = 1$

$\therefore \quad u + iv = \log r + i\theta$

$\Rightarrow \quad u = \log r$ and $v = \theta$ in the polar form.

Since we know that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$

$u = \log\sqrt{x^2 + y^2}$ and $v = \tan^{-1}(y/x)$ in the cartesian form.

Limit : A complex valued function $f(z)$ defined in the neighbourhood of a point z_0 is said to have a *limit* l as z tends to z_0, if for every $\varepsilon > 0$ however small there exists a positive real number δ such that $|f(z) - l| < \varepsilon$ when $|z - z_0| < \delta$. This is written as $\lim\limits_{z \to z_0} f(z) = l$

Continuity : A complex valued function $f(z)$ is said to be *continuous* at $z = z_0$ if $f(z_0)$ exists and $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

That is to say that $|f(z) - f(z_0)| < \varepsilon$ when $|z - z_0| < \delta$.

Differentiability : A complex valued function $f(z)$ is said to be differentiable at $z = z_0$ if $\lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists and is unique. This limit when exists is called the *derivative* of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$.

Suppose we write $\delta z = z - z_0$, then $z \to z_0$ implies that $\delta z \to 0$

Hence, $f'(z_0) = \lim\limits_{\delta z \to 0} \dfrac{f(z_0+\delta z)-f(z_0)}{\delta z}$

Further $f(z)$ is said to be continuous / differentiable in a domain or a region D if $f(z)$ is continuous / differentiable at every point of D.

These definitions are analogous with the definitions of a real valued function.

3.4 Analytic Function and Connected Theorems

A complex valued function $w = f(z)$ is said to be *analytic* at a point $z = z_0$ if $\dfrac{dw}{dz} = f'(z) = \mathop{lt}\limits_{\delta z \to 0} \dfrac{f(z+\delta z)-f(z)}{\delta z}$ exists and is unique at z_0 and in the neighbourhood of z_0. Further $f(z)$ is said to be analytic in a region if it is analytic at every point of the region.

Analytic function is also called a *regular function or holomorphic function*. We can as well say that $f(z)$ is analytic at a point z_0 if it is differentiable at z_0 and in the neighbourhood of z_0.

3.41 Theorem-1 [Cauchy-Riemann equations in the cartesian form]

The necessary conditions that the function $w = f(z) = u(x, y) + i\,v(x, y)$ may be analytic at any point $z = x + iy$ is that, there exists four continuous first order partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ and satisfy the equations : $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ These are known as Cauchy-Riemann (C – R) equations : $u_x = v_y$ and $v_x = -u_y$

Proof : Let $f(z)$ be analytic at a point $z = x + i\,y$ and hence by the definition,

$$f'(z) = \mathop{lt}\limits_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z} \text{ exists and is unique.}$$

In the cartesian form $f(z) = u(x, y) + i\,v(x, y)$ and let δz be the increment in z corresponding to the increments δx, δy in x, y.

$$f'(z) = \underset{\delta z \to 0}{lt} \frac{[u(x+\delta x, y+\delta y)+iv(x+\delta x, y+\delta y)]-[u(x, y)+iv(x, y)]}{\delta z}$$

$$f'(z) = \underset{\delta z \to 0}{lt} \frac{[u(x+\delta x, y+\delta y)-u(x, y)]}{\delta z}$$

$$+i \underset{\delta z \to 0}{lt} \frac{[v(x+\delta x, y+\delta y)-v(x, y)]}{\delta z} \quad \ldots (1)$$

Now $\delta z = (z+\delta z)-z$ where $z = x+iy$

$\therefore$ $\delta z = [(x+\delta x)+i(y+\delta y)]-[x+iy]$

i.e., $\delta z = \delta x+i\delta y$

Since δz tends to zero, we have the following two possibilities.

Case (i) : Let $\delta y = 0$ so that $\delta z = \delta x$ and $\delta z \to 0$ imply $\delta x \to 0$.

Now (1) becomes

$$f'(z) = \underset{\delta x \to 0}{lt} \frac{u(x+\delta x, y)-u(x, y)}{\delta x} + i \underset{\delta x \to 0}{lt} \frac{v(x+\delta x, y)-v(x, y)}{\delta x}$$

These limits from the basic definition are the partial derivatives of u and v w.r.t. x.

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \ldots (2)$$

Case (ii) : Let $\delta x = 0$ sothat $\delta z = i\delta y$ and $\delta z \to 0$ imply $i\delta y \to 0$ or $\delta y \to 0$.

Now (1) becomes

$$f'(z) = \underset{\delta y \to 0}{lt} \frac{u(x, y+\delta y)-u(x, y)}{i\delta y} + i \underset{\delta y \to 0}{lt} \frac{v(x, y+\delta y)-v(x, y)}{i\delta y}$$

But $1/i = i/i^2 = i/-1 = -i$ and hence we have,

$$f'(z) = \underset{\delta y \to 0}{lt} -i \cdot \frac{u(x, y+\delta y)-u(x, y)}{\delta y} + \underset{\delta y \to 0}{lt} \frac{v(x, y+\delta y)-v(x, y)}{\delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore \quad f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \ldots (3)$$

Equating the RHS of (2) and (3) we have,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Now equating the real and imaginary parts we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Thus we have established **Cauchy-Riemann equations :** $u_x = v_y$ **and** $v_x = -u_y$

These are the necessary conditions in the cartesian form for the complex valued function $f(z) = u + iv$ to be analytic.

3.42 Theorem-2 [Cauchy-Riemann equations in the polar form]

If $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the equations :

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad and \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

These are known as Cauchy-Riemann (C – R) equations in the polar form.

Proof : Let $f(z)$ be analytic at a point $z = re^{i\theta}$ and hence by definition,

$$f'(z) = \underset{\delta z \to 0}{lt} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exists and is unique.}$$

In the polar form $f(z) = u(r, \theta) + iv(r, \theta)$ and let δz be the increment in z corresponding to the increments δr, $\delta\theta$ in r, θ

$$f'(z) = \underset{\delta z \to 0}{lt} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r, \theta) + iv(r, \theta)]}{\delta z}$$

$$f'(z) = \underset{\delta z \to 0}{lt} \frac{u(r+\delta r, \theta+\delta\theta) - u(r, \theta)}{\delta z}$$

$$+ i \underset{\delta z \to 0}{lt} \frac{v(r+\delta r, \theta+\delta\theta) - v(r, \theta)}{\delta z} \quad \ldots (1)$$

Consider $z = re^{i\theta}$. Since z is a function of two variables r, θ we have,

$$\delta z = \frac{\partial z}{\partial r}\,\delta r + \frac{\partial z}{\partial \theta}\,\delta\theta$$

$$= \frac{\partial (r e^{i\theta})}{\partial r}\,\delta r + \frac{\partial (r e^{i\theta})}{\partial \theta}\,\delta\theta$$

i.e., $\delta z = e^{i\theta}\,\delta r + i r e^{i\theta}\,\delta\theta$

Since δz tends to zero, we have the following two possibilities.

Case (i) : Let $\delta\theta = 0$ so that $\delta z = e^{i\theta}\delta r$ and $\delta z \to 0$ imply $\delta r \to 0$.
Now (1) becomes,

$$f'(z) = \underset{\delta r \to 0}{lt}\ \frac{u(r+\delta r,\ \theta) - u(r,\ \theta)}{e^{i\theta}\,\delta r} + i\ \underset{\delta r \to 0}{lt}\ \frac{v(r+\delta r,\ \theta) - v(r,\ \theta)}{e^{i\theta}\,\delta r}$$

i.e., $$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \qquad \ldots (2)$$

Case (ii) : Let $\delta r = 0$ so that $\delta z = i r e^{i\theta}\delta\theta$ and $\delta z \to 0$ imply $\delta\theta \to 0$.
Now (1) becomes

$$f'(z) = \underset{\delta\theta \to 0}{lt}\ \frac{u(r,\ \theta+\delta\theta) - u(r,\ \theta)}{i r e^{i\theta}\,\delta\theta} + i\ \underset{\delta\theta \to 0}{lt}\ \frac{v(r,\ \theta+\delta\theta) - v(r,\ \theta)}{i r e^{i\theta}\,\delta\theta}$$

$$= \frac{1}{i r e^{i\theta}}\left[\underset{\delta\theta \to 0}{lt}\ \frac{u(r,\ \theta+\delta\theta) - u(r,\ \theta)}{\delta\theta} + i\ \underset{\delta\theta \to 0}{lt}\ \frac{v(r,\ \theta+\delta\theta) - v(r,\ \theta)}{\delta\theta}\right]$$

$$f'(z) = \frac{1}{i r e^{i\theta}}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right] = \frac{1}{r e^{i\theta}}\left[\frac{1}{i}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right]$$

But $1/i = i/i^2 = i/-1 = -i$ and hence we have,

$$f'(z) = \frac{1}{r e^{i\theta}}\left[-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right] = e^{-i\theta}\left[\frac{-i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right]$$

ie., $$f'(z) = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right] \qquad \ldots (3)$$

Equating the RHS of (2) and (3) we have,

$$e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right]$$

Cancelling $e^{-i\theta}$ on both sides and equating the real and imaginary parts we get,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \textbf{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \textbf{or} \quad r u_r = v_\theta \quad \textbf{and} \quad r v_r = -u_\theta$$

These are **Cauchy-Riemann equations in the polar form.**

3.5 Properties of Analytic Functions

Harmonic function - Definition

A function ϕ is said to be ***harmonic*** if it satisfies Laplace's equation $\nabla^2 \phi = 0$.

In the cartesian form $\phi(x, y)$ is harmonic if $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$

In the polar form $\phi(r, \theta)$ is harmonic if $\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = 0$

3.51 Harmonic property

The real and imaginary parts of an analytic function are harmonic.

Proof : We shall prove the result separately for the cartesian and polar form of z.

Cartesian form

Let $f(z) = u(x, y) + iv(x, y)$ be analytic. We shall show that u and v satisfy Laplace's equation in the cartesian form

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Since $f(z)$ is analytic we have Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \ldots (1)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \ldots (2)$$

Differentiating (1) w.r.t. x and (2) w.r.t. y partially we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y}, \quad \frac{\partial^2 v}{\partial y\, \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

But $\dfrac{\partial^2 v}{\partial x\, \partial y} = \dfrac{\partial^2 v}{\partial y\, \partial x}$ is always true and hence we have

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \text{ or } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \text{ is harmonic.}$$

Again differentiating (1) w.r.t y and (2) w.r.t. x partially we get,

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 v}{\partial y^2}, \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x\, \partial y}$$

But $\dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{\partial^2 u}{\partial x\, \partial y}$ is always true and hence we have

$$\frac{\partial^2 v}{\partial y^2} = \frac{-\partial^2 v}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow v \text{ is harmonic.}$$

Thus we have proved that the real and imaginary parts of an analytic function when expressed in the cartesian form satisfy Laplace's equation in the cartesian form.

Polar form

Let $f(z) = u(r, \theta) + i\, v(r, \theta)$ be analytic. We shall show that u and v satisfy Laplace's equation in the polar form

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

We have Cauchy-Riemann equations in the polar form

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad \ldots (1)$$

$$r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} \qquad \ldots (2)$$

Differentiating (1) w.r.t. r and (2) w.r.t. θ partially we get,

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\cdot 1 = \frac{\partial^2 v}{\partial r\, \partial \theta}\ ;\ r\frac{\partial^2 v}{\partial \theta\, \partial r} = -\frac{\partial^2 u}{\partial \theta^2}$$

But $\dfrac{\partial^2 v}{\partial r\, \partial \theta} = \dfrac{\partial^2 v}{\partial \theta\, \partial r}$ is always true and hence we have

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{-1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

Dividing by r and transposing the term in the RHS to LHS we obtain

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$\therefore$ u satisfies Laplace's equation in the polar form $\Rightarrow$ **u is harmonic.**

Again differentiating (1) w.r.t θ and (2) w.r.t. r partially we get

$$r\frac{\partial^2 u}{\partial \theta\, \partial r} = \frac{\partial^2 v}{\partial \theta^2}\ ;\ r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r}\cdot 1 = \frac{-\partial^2 u}{\partial r\, \partial \theta}$$

But $\dfrac{\partial^2 u}{\partial \theta\, \partial r} = \dfrac{\partial^2 u}{\partial r\, \partial \theta}$ is always true and hence we have

$$\frac{1}{r}\frac{\partial^2 v}{\partial \theta^2} = -\left(r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r}\right)$$

Dividing by r and transposing terms in the RHS to LHS we obtain

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

$\therefore$ v satisfies Laplace's equation in the polar form $\Rightarrow$ **v is harmonic.**

Thus we have proved that u and v are harmonic.

Note : *The converse of this theorem is not true. That is to say that we can give examples of function u & v satisfying Laplace's equation but not satisfying Cauchy-Riemann equations.*

Let $u = x^2 - y^2$, $\qquad v = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 3x^2 - 3y^2,\ \frac{\partial v}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 2,\ \frac{\partial^2 u}{\partial y^2} = -2 \qquad \frac{\partial^2 v}{\partial x^2} = 6x,\ \frac{\partial^2 v}{\partial y^2} = -6x$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0. \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6x - 6x = 0$$

This shows that u and v are harmonic functions.

But Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$ are not satisfied.

Hence $u + iv$ is not analytic.

3.52 Orthogonal Property

If $f(z) = u + iv$ is analytic then the family of curves $u(x, y) = c_1$, $v(x, y) = c_2$, c_1 and c_2 being constants, intersect each other orthogonally.

Proof : We know that two cuves intersect each other orthogonally if the tangents at the point of intersection are at right angles. Further we know that $\frac{dy}{dx}$ represents slope of the tangent and the condition for perpendicularity of two lines is that the product of their slopes must be equal to -1.

Consider $u(x, y) = c_1$ and differentiating w.r.t x treating y as a function of x we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1 \ (say)$$

Similarly for $v(x, y) = c_2$, $\frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = m_2$ (*say*)

$$\therefore \quad m_1 m_2 = \frac{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}} \qquad \ldots (1)$$

But $f(z) = u + iv$ is analytic and hence we have Cauchy-Riemann equations : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Using these in (1) we have,

$$m_1 m_2 = \frac{\frac{\partial v}{\partial y} \cdot \frac{-\partial u}{\partial y}}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}} = -1$$

Hence the curves intersect orthogonally at every point of intersection.

Note : *Converse of this theorem is not true and it is illustrated by the following example.*

Let $\quad u = \frac{x^2}{y}$ and $v = x^2 + 2y^2$

We shall show that the curves $u = c_1$ and $v = c_2$ (c_1 *and* c_2 *being constants*) intersect orthogonally but u and v does not satisfy Cauchy-Riemann equations.

Consider $\quad \frac{x^2}{y} = c_1 \ ; \ x^2 + 2y^2 = c_2$

Differentiating these w.r.t. x treating y as a function of x we obtain

$$\frac{y(2x) - x^2 \cdot \frac{dy}{dx}}{y^2} = 0 \qquad ; \ 2x + 4y\frac{dy}{dx} = 0$$

i.e., $\quad 2xy - x^2\frac{dy}{dx} = 0 \qquad ; \ 4y\frac{dy}{dx} = -2x$

$$\therefore \quad \frac{dy}{dx} = \frac{2xy}{x^2} = \frac{2y}{x} = m_1 \ (say) \ ; \ \frac{dy}{dx} = -\frac{x}{2y} = m_2 \ (say)$$

Now $\quad m_1 \cdot m_2 = \frac{2y}{x} \cdot \frac{-x}{2y} = -1$

Hence $u = c_1$ and $v = c_2$ intersect orthogonally.

Further we have, $\frac{\partial u}{\partial x} = \frac{2x}{y}$, $\frac{\partial u}{\partial y} = \frac{-x^2}{y^2}$, $\frac{\partial v}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 4y$

Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ are not satisfied.

Thus we conclude that $u + iv$ is not analytic.

Note : *The result can also be established for the polar family of curves.*

If $r = f(\theta)$ we know that $\tan\phi = r\frac{d\theta}{dr}$, ϕ being the angle between the radius vector and the tangent. The angle between the tangents at the point of intersection of the curves is $\phi_1 - \phi_2$ and $\tan\phi_1 \cdot \tan\phi_2 = -1$ is the condition for orthogonality.

Consider $u(r, \theta) = c_1$ and differentiate w.r.t. θ treating r as a function of θ.

$$\therefore \quad u_r \frac{dr}{d\theta} + u_\theta = 0 \text{ or } \frac{dr}{d\theta} = \frac{-u_\theta}{u_r}$$

Hence $\tan\phi_1 = r\frac{d\theta}{dr} = -\frac{r u_r}{u_\theta}$

Similarly for the curve $v(r, \theta) = c_2$, $\tan\phi_2 = \frac{-r v_r}{v_\theta}$

$$\therefore \quad \tan\phi_1 \cdot \tan\phi_2 = \frac{(r u_r)(r v_r)}{u_\theta \cdot v_\theta}$$

But $\quad r u_r = v_\theta$ and $r v_r = -u_\theta$ by $C-R$ equations.

Now $\quad \tan\phi_1 \cdot \tan\phi_2 = \frac{(v_\theta)\cdot(-u_\theta)}{u_\theta \cdot v_\theta} = -1$

Thus the polar family of curves $u(r, \theta) = c_1$ and $v(r, \theta) = c_2$ intersect each other orthogonally.

WORKED PROBLEMS

Type-1 : ***Finding the derivative of an analytic function.***

Working procedure for problems

- Given $w = f(z)$, we substitute $z = x + iy$ or $z = re^{i\theta}$ to find the real and imaginary parts u and v as functions of x, y or r, θ.
- We find first order partial derivatives and verify Cauchy-Riemann equations in the cartesian or polar form to conclude that $f(z)$ is analytic.

- To find the derivative of $f(z)$ we make use of the fundamental results derived while establishing $C-R$ equations. They are as follows.

$$f'(z) = u_x + i v_x \qquad \text{[Cartesian form]}$$

$$f'(z) = e^{-i\theta}(u_r + i v_r) \qquad \text{[Polar form]}$$

- We substitute for the partial derivatives and rearrange as a function of $(x+iy)$ or $re^{i\theta}$ which is z, with the result $f'(z)$ is obtained as a function of z.

---

1. *Show that* $w = z + e^z$ *is analytic and hence find* $\dfrac{dw}{dz}$

>> By data $w = z + e^z$

i.e., $u + iv = (x+iy) + e^{(x+iy)}$

$= (x+iy) + e^x \cdot e^{iy}$

$= (x+iy) + e^x(\cos y + i\sin y)$

i.e., $u + iv = (x + e^x \cos y) + i(y + e^x \sin y)$

$\therefore \quad u = x + e^x \cos y \qquad v = y + e^x \sin y$

$u_x = 1 + e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = 1 + e^x \cos y$

We observe that Cauchy-Riemann equations in the cartesian form $u_x = v_y$ and $v_x = -u_y$ are satisfied.

Thus $w = z + e^z$ **is analytic**

Also we have $\dfrac{dw}{dz} = f'(z) = u_x + i v_x$

i.e., $\dfrac{dw}{dz} = (1 + e^x \cos y) + i(e^x \sin y)$

$= 1 + e^x(\cos y + i\sin y) = 1 + e^x \cdot e^{iy} = 1 + e^{x+iy}$

Since $z = x + iy$, $\mathbf{\dfrac{dw}{dz} = 1 + e^z}$

---

2. *Show that* $f(z) = \sin z$ *is analytic and hence find* $f'(z)$.

>> By data $f(z) = \sin z$

i.e., $u + iv = \sin(x + iy)$

$= \sin x \cos iy + \cos x \sin iy$

i.e., $u + iv = \sin x \cosh y + i \cos x \sinh y$

[*We have used* $\sin(A + B) = \sin A \cos B + \cos A \sin B$ *and the results* $\sin(i\theta) = i \sinh \theta$, $\cos(i\theta) = \cosh \theta$]

$\therefore$ $u = \sin x \cosh y$ $\quad v = \cos x \sinh y$

$u_x = \cos x \cosh y$ $\quad v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $\quad v_y = \cos x \cosh y$

Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ are satisfied.

Thus $f(z) = \sin z$ **is analytic**.

Also we have $f'(z) = u_x + i v_x$

i.e., $f'(z) = \cos x \cosh y + i(-\sin x \sinh y)$

Using $\cosh y = \cos iy$ and $i \sinh y = \sin(iy)$ we have,

$$f'(z) = \cos x \cos iy - \sin x \sin(iy) = \cos(x + iy)$$

[*We have used* $\cos A \cos B - \sin A \sin B = \cos(A + B)$]

Since $z = x + iy$, $f'(z) = \cos z$

---

3. *Show that* $f(z) = \cosh z$ *is analytic and hence find* $f'(z)$

>> By data $f(z) = \cosh z$

i.e., $u + iv = \cosh(x + iy)$

$= \cos i(x + iy)$, since $\cosh \theta = \cos i\theta$

$= \cos(ix - y)$

$= \cos ix \cos y + \sin ix \sin y$

i.e., $u + iv = \cosh x \cos y + i \sinh x \sin y$

$\therefore$ $u = \cosh x \cos y$ $\quad v = \sinh x \sin y$

$u_x = \sinh x \cos y$ $\quad v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y$ $\quad v_y = \sinh x \cos y$

Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ are satisfied.

Thus $f(z) = \cosh z$ **is analytic.**

Also we have $f'(z) = u_x + i v_x$

i.e., $f'(z) = \sinh x \cos y + i \cosh x \sin y$

Multiplying and dividing by i in the RHS we have,

$$f'(z) = \frac{1}{i} [i \sinh x \cos y - \cosh x \sin y]$$

$$= \frac{1}{i} [\sin ix \cos y - \cos ix \sin y]$$

$$= \frac{1}{i} \cdot \sin (ix - y) = \frac{1}{i} \sin i (x + iy)$$

i.e., $f'(z) = \frac{1}{i} \cdot i \sinh (x + iy) = \sinh (x + iy)$

Since $z = x + iy$, $f'(z) = \sinh z$

Note : *Finding u and v from $f(z)$ and later finding $f'(z)$ as a function of z can also be done in the following alternative manner.*

$$f(z) = \cosh z = \cosh (x + iy)$$

ie., $u + iv = \frac{1}{2} \left[e^{x+iy} + e^{-(x+iy)} \right]$

$$= \frac{1}{2} \left[e^x (\cos y + i \sin y) + e^{-x} (\cos y - i \sin y) \right]$$

$$= \frac{1}{2} (e^x + e^{-x}) \cos y + i \cdot \frac{1}{2} (e^x - e^{-x}) \sin y$$

ie., $u + iv = \cosh x \cos y + i \sinh x \sin y$

$\Rightarrow$ $u = \cosh x \cos y$ and $v = \sinh x \sin y$

Also $f'(z) = u_x + i v_x$

$$= \sinh x \cos y + i \cosh x \sin y$$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} + i \cdot \frac{e^x + e^{-x}}{2} \cdot \frac{e^{iy} - e^{-iy}}{2i}$$

$$= \frac{1}{4} \left[2 e^{x+iy} - 2 e^{-(x+iy)} \right] = \frac{1}{2} \left[e^{x+iy} - e^{-(x+iy)} \right]$$

Thus $f'(z) = \sinh (x + iy) = \sinh z$

4. *Show that* $w = \log z$, $z \neq 0$ *is analytic and hence find* $\dfrac{dw}{dz}$

>> *[It is convenient to do the problem in the polar form as* u *and* v *can be found easily]*

By data $w = \log z$. Taking $z = r e^{i\theta}$ we have,

$$u + iv = \log(re^{i\theta}) = \log r + \log(e^{i\theta}) = \log r + i\theta \log_e e$$

i.e., $u + iv = \log r + i\theta \quad (\because \log_e e = 1)$

$\therefore \quad u = \log r \qquad v = \theta$ and hence we have,

$$u_r = \frac{1}{r} \qquad v_r = 0$$

$$u_\theta = 0 \qquad v_\theta = 1$$

$C-R$ equations in the polar form : $r u_r = v_\theta$ and $r v_r = -u_\theta$ are satisfied.

Thus $w = \log z$ **is analytic**.

Also we have in the polar form,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left(\frac{1}{r} + i \cdot 0\right) = \frac{1}{re^{i\theta}}$$

Since $z = re^{i\theta}$, $f'(z) = \dfrac{1}{z}$

---

5. *Show that* $f(z) = z^n$, *where* n *is a positive integer is analytic and hence find its derivative.*

>> By data $f(z) = z^n$. Taking $z = re^{i\theta}$ we have,

$$u + iv = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

$\therefore \quad u = r^n \cos n\theta \qquad v = r^n \sin n\theta$

$$u_r = n r^{n-1}\cos n\theta \qquad v_r = n r^{n-1}\sin n\theta$$

$$u_\theta = -n r^n \sin n\theta \qquad v_\theta = n r^n \cos n\theta$$

C-R equations in the polar form $r u_r = v_\theta$ and $r v_r = -u_\theta$ are satisfied.

Thus $f(z) = z^n$ **is analytic**.

Also we have $f'(z) = e^{-i\theta}(u_r + i v_r)$

i.e., $f'(z) = e^{-i\theta}(n r^{n-1}\cos n\theta + i n r^{n-1}\sin n\theta)$

$= n r^{n-1} e^{-i\theta}(\cos n\theta + i\sin n\theta)$

$= n r^{n-1} e^{-i\theta}\cdot e^{in\theta}$

$= n r^{n-1} e^{i\theta(n-1)} = n r^{n-1}(e^{i\theta})^{n-1} = n(re^{i\theta})^{n-1}$

Since $z = re^{i\theta}$, $f'(z) = n z^{n-1}$

---

6. *Show that* $w = z\bar{z}$ *is not analytic.*

>> By data $w = z\bar{z}$. Since $\bar{z} = x - iy$ we have,

$u + iv = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2+y^2$

$\therefore$ $u = x^2+y^2$; $v = 0$

$u_x = 2x,\ u_y = 2y$; $v_x = 0,\ v_y = 0$

Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ are not satisfied.

Thus $w = z\bar{z}$ **is not analytic.**

---

7. *Show that* $f(z) = e^x(\cos y + i\sin y)$ *is holomorphic.*

>> By data $f(z) = e^x(\cos y + i\sin y)$

i.e., $u + iv = (e^x\cos y) + i(e^x\sin y)$

$\therefore$ $u = e^x\cos y$ $\quad v = e^x\sin y$

$u_x = e^x\cos y$ $\quad v_x = e^x\sin y$

$u_y = -e^x\sin y$ $\quad v_y = e^x\cos y$

$C-R$ equations $u_x = v_y$ and $v_x = -u_y$ are satisfied.

Thus the given $f(z)$ **is holomorphic**.

---

8. *Find the derivative of the analytic function* z^z

>> $f(z) = u + iv = z^z$

$\therefore$ $\log(u+iv) = z\log z = re^{i\theta}\log(re^{i\theta}) = (re^{i\theta})(\log r + i\theta)$

i.e., $\log(u+iv) = r(\cos\theta + i\sin\theta)(\log r + i\theta)$

$$\log(u+iv) = (r\log r\cos\theta - r\theta\sin\theta) + i(r\log r\sin\theta + r\theta\cos\theta)$$

Let $\log(u+iv) = A+iB$ (*say*) ... (1)

where we have

$$A = r\log r\cos\theta - r\theta\sin\theta \quad \dots (2)$$

$$B = r\log r\sin\theta + r\theta\cos\theta \quad \dots (3)$$

From (1) $u+iv = e^{A+iB} = e^A\cos B + ie^A\sin B$

$\therefore$ $u = e^A\cos B$ and $v = e^A\sin B$ where A and B are functions of r and θ

Now
$$\frac{\partial u}{\partial r} = -e^A\sin B\,\frac{\partial B}{\partial r} + e^A\,\frac{\partial A}{\partial r}\cos B$$

$$= e^A\left(\frac{\partial A}{\partial r}\cos B - \frac{\partial B}{\partial r}\sin B\right)$$

$$\frac{\partial v}{\partial r} = e^A\cos B\,\frac{\partial B}{\partial r} + e^A\,\frac{\partial A}{\partial r}\sin B = e^A\left(\frac{\partial B}{\partial r}\cos B + \frac{\partial A}{\partial r}\sin B\right)$$

$$\therefore \quad \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = e^A\left[\frac{\partial A}{\partial r}(\cos B + i\sin B) + \frac{\partial B}{\partial r}(-\sin B + i\cos B)\right]$$

$$= e^A\left[\frac{\partial A}{\partial r}(\cos B + i\sin B) + i\frac{\partial B}{\partial r}(\cos B + i\sin B)\right]$$

i.e.,
$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = e^{A+iB}\left[\frac{\partial A}{\partial r} + i\frac{\partial B}{\partial r}\right] \quad \dots (4)$$

Now from (2) and (3) we obtain

$$\frac{\partial A}{\partial r} = (1+\log r)\cos\theta - \theta\sin\theta \; ; \; \frac{\partial B}{\partial r} = (1+\log r)\sin\theta + \theta\cos\theta$$

$$\therefore \quad \frac{\partial A}{\partial r} + i\frac{\partial B}{\partial r} = (1+\log r)(\cos\theta + i\sin\theta) - \theta\sin\theta + i\theta\cos\theta$$

$$= (1+\log r)\,e^{i\theta} + i\theta(\cos\theta + i\sin\theta)$$

$$= e^{i\theta}\,[1+\log r + i\theta] = e^{i\theta}\left[1+\log r + \log e^{i\theta}\right]$$

i.e.,
$$\frac{\partial A}{\partial r} + i\frac{\partial B}{\partial r} = e^{i\theta}\left[1+\log(re^{i\theta})\right] = e^{i\theta}(1+\log z) \quad \dots (5)$$

Using (5) in (4) we obtain,

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = e^{A+iB}\cdot e^{i\theta}\,(1+\log z)$$

Further $e^{A+iB} = u + i\,v = z^z$

$$\therefore \quad \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = e^{i\theta} \cdot z^z (1 + \log z)$$

or $$e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = z^z(1 + \log z)$$

But $$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Thus we have $\mathbf{f'(z) = z^z(1 + \log z)}$

---

9. *Given that* $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

show that $f(z)$ *satisfies Cauchy - Riemann equations at the origin.*

>> We have $f(z) = u + i\,v = \dfrac{x^3 - y^3}{x^2+y^2} + i\,\dfrac{x^3+y^3}{x^2+y^2}$ for $z \neq 0$

$\Rightarrow \quad u = \dfrac{x^3-y^3}{x^2+y^2}$ and $v = \dfrac{x^3+y^3}{x^2+y^2}$ where $(x, y) \neq (0, 0)$

Also $u(0, 0) = 0 = v(0, 0)$ since $f(z) = 0$ when $z = 0$

$u_x,\ u_y,\ v_x,\ v_y$ becomes indeterminate at $(0, 0)$ and hence we shall find the same from the basic definition.

We have by the basic definition

$$u_x = \frac{\partial u}{\partial x} = \underset{\delta x \to 0}{lt}\ \frac{u(x+\delta x, y) - u(x, y)}{\delta x}$$

$$u_y = \frac{\partial u}{\partial y} = \underset{\delta y \to 0}{lt}\ \frac{u(x, y+\delta y) - u(x, y)}{\delta y}$$

Similarly we can write for v_x and v_y also.

Now, $$[u_x]_{(0,0)} = \underset{\delta x \to 0}{lt}\ \frac{u(\delta x, 0) - u(0, 0)}{\delta x}$$

$$= \underset{\delta x \to 0}{lt}\ \frac{(\delta x)^3/(\delta x)^2 - 0}{\delta x} = 1$$

$$[u_y]_{(0,0)} = \lim_{\delta y \to 0} \frac{u(0, \delta y) - u(0, 0)}{\delta y}$$

$$= \lim_{\delta y \to 0} \frac{-(\delta y)^3/(\delta y)^2 - 0}{\delta y} = -1$$

Similarly we also have,

$$[v_x]_{(0,0)} = \lim_{\delta x \to 0} \frac{v(\delta x, 0) - v(0, 0)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\delta x)^3/(\delta x)^2 - 0}{\delta x} = 1$$

$$[v_y]_{(0,0)} = \lim_{\delta y \to 0} \frac{v(0, \delta y) - v(0, 0)}{\delta y}$$

$$= \lim_{\delta y \to 0} \frac{(\delta y)^3/(\delta y)^2 - 0}{\delta y} = 1$$

We observe that $u_x = v_y$ and $v_x = -u_y$ at the origin.

This shows that $f(z)$ **satisfies Cauchy - Riemann equations at the origin.**

---

10. *Show that* $f(z) = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$, $r \neq 0$ *is a regular function of* $z = re^{i\theta}$. *Also find* $f'(z)$.

>> We have $u = \left(r + \frac{k^2}{r}\right)\cos\theta$; $v = \left(r - \frac{k^2}{r}\right)\sin\theta$

$$\therefore \quad u_r = \left(1 - \frac{k^2}{r^2}\right)\cos\theta \qquad v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$$

$$u_\theta = -\left(r + \frac{k^2}{r}\right)\sin\theta \qquad v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$$

Hence $r\,u_r = \left(r - \frac{k^2}{r}\right)\cos\theta$ and $r\,v_r = \left(r + \frac{k^2}{r}\right)\sin\theta$

$C-R$ equations $r\,u_r = v_\theta$ and $r\,v_r = -u_\theta$ are satisfied.

Thus $f(z)$ **is analytic.**

We have $f'(z) = e^{-i\theta}(u_r + i v_r)$

$$f'(z) = e^{-i\theta}\left[\left(1-\frac{k^2}{r^2}\right)\cos\theta + i\left(1+\frac{k^2}{r^2}\right)\sin\theta\right]$$

$$= e^{-i\theta}\left[(\cos\theta + i\sin\theta) - \frac{k^2}{r^2}(\cos\theta - i\sin\theta)\right]$$

$$= e^{-i\theta}\left[e^{i\theta} - \frac{k^2}{r^2}e^{-i\theta}\right] = 1 - \frac{k^2}{(r e^{i\theta})^2}$$

Since $z = r e^{i\theta}$ we get $f'(z) = 1 - (k^2/z^2)$

Type-2 : ***Construction of analytic function*** $f(z)$ ***given its real or imaginary part.***

Working procedure for problems

- Given u or v as functions of x, y we find u_x, u_y or v_x, v_y and consider $f'(z) = u_x + i v_x$
- Given u, we use C-R equation $v_x = -u_y$ or given v we use $u_x = v_y$ so that $f'(z) = u_x - i u_y$ or $f'(z) = v_y + i v_x$
- We substitute the expression for the partial derivatives in the RHS and then put $x = z, y = 0$ to obtain $f'(z)$ as a function of z.
- Integrating w.r.t. z we get $f(z)$.
- Similarly in the case of polar co-ordinates r, θ we consider $f'(z) = e^{-i\theta}(u_r + i v_r)$ and use C – R equation in the RHS, $v_r = \frac{-1}{r} u_\theta$ given u or $u_r = \frac{1}{r} v_\theta$ given v.
- We use the substitution $r = z, \theta = 0$ to obtain $f'(z)$ as a function of z.
- Integrating w.r.t z we get $f(z)$.

This method is known as Milne-Thompson method.

Remark: *Substitution of $x = z$ and $y = 0$; $r = z$ and $\theta = 0$ known as Milne-Thompson substitution instantly converts the RHS into a function of z. This will be highly helpful when the RHS is in a complicated form. However we can plan to convert RHS which is a function of x and y into a function of $x + iy = z$ with the result we get $f'(z)$ as a function of z. Similarly in the case of polar coordinates we can plan to convert the RHS which is a function of r and θ into a funciton of $r e^{i\theta} = z$ with the result we get $f'(z)$ as a function of z.*

$f(z)$ is obtained on integration w.r.t. z in both the cases.

11. *Construct the analytic function whose real part is* $u = \log\sqrt{x^2+y^2}$

>> $u = \log\sqrt{x^2+y^2} = \log(x^2+y^2)^{1/2} = \frac{1}{2}\log(x^2+y^2)$

$$\therefore \quad u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$u_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}$$

Consider $f'(z) = u_x + i\,v_x$. But $v_x = -u_y$ (*C – R equation.*)

$$\therefore \quad f'(z) = u_x - i\,u_y = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \qquad \ldots(1)$$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = \frac{z}{z^2+0} - i\cdot 0 = \frac{1}{z}$$

$$\therefore \quad f(z) = \int \frac{1}{z}\,dz + c$$

Thus $\mathbf{f(z) = \log z + c}$

Remark : *Referring to (1) we have,*

$$f'(z) = \frac{x-iy}{x^2+y^2} = \frac{(x-iy)}{(x-iy)(x+iy)} = \frac{1}{x+iy}$$

$$\therefore \quad f'(z) = \frac{1}{x+iy} = \frac{1}{z} \quad (\textit{without using } x = z,\ y = 0)$$

Hence $\mathbf{f(z) = \log z + c}$

---

12. *Find the analytic function* $f(z)$ *whose imaginary part is* $e^x(x\sin y + y\cos y)$

>> By data $v = e^x(x\sin y + y\cos y)$

$$\therefore \quad v_x = e^x(\sin y) + (x\sin y + y\cos y)e^x \qquad [\textit{By product rule}]$$

i.e., $v_x = e^x(\sin y + x\sin y + y\cos y)$ ...(1)

Also $v_y = e^x(x\cos y - y\sin y + \cos y)$ ...(2)

Consider $f'(z) = u_x + i\,v_x$. But $u_x = v_y$ (*C-R equation*)

i.e., $f'(z) = v_y + i\,v_x$

$$= e^x(x\cos y - y\sin y + \cos y) + i\,e^x(\sin y + x\sin y + y\cos y)$$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = e^z(z+1) \quad \text{since} \quad \sin 0 = 0,\ \cos 0 = 1$$

$$\therefore \quad f(z) = \int (z+1)e^z\, dz + c$$

Integrating by parts,

$$f(z) = (z+1)e^z - \int e^z \cdot 1\, dz + c = (z+1)e^z - e^z + c$$

Thus $\boldsymbol{f(z) = z e^z + c}$

---

13. *Find the analytic function whose real part is* $\dfrac{x^4 - y^4 - 2x}{x^2 + y^2}$ *Hence determine* v.

>> $$u = \frac{x^4 - y^4 - 2x}{x^2 + y^2} \text{ by data.}$$

$$\therefore \quad u_x = \frac{(x^2 + y^2)(4x^3 - 2) - (x^4 - y^4 - 2x)2x}{(x^2 + y^2)^2}$$

$$u_y = \frac{(x^2 + y^2)(-4y^3) - (x^4 - y^4 - 2x)2y}{(x^2 + y^2)^2}$$

Consider $f'(z) = u_x + i v_x$. But $v_x = -u_y$ (C-R equation)

$$\therefore \quad f'(z) = u_x - i u_y$$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = [u_x]_{(z,0)} - i\,[u_y]_{(z,0)}$$

i.e., $$f'(z) = \frac{z^2(4z^3 - 2) - (z^4 - 2z)2z}{(z^2)^2} - i(0)$$

$$= \frac{4z^5 - 2z^2 - 2z^5 + 4z^2}{z^4} = \frac{2z^5 + 2z^2}{z^4}$$

i.e., $$f'(z) = 2\frac{z^5}{z^4} + 2\frac{z^2}{z^4} = 2z + \frac{2}{z^2}$$

$$\therefore \quad f(z) = \int \left(2z + \frac{2}{z^2}\right) dz + c = z^2 - \frac{2}{z} + c$$

Thus $\boldsymbol{f(z) = z^2 - \dfrac{2}{z} + c}$

To find v, we shall separate the RHS of $f(z)$ into real and imaginary parts.

i.e., $u + iv = (x+iy)^2 - \dfrac{2}{x+iy} + c$

$$= (x^2 + i^2y^2 + 2xiy) - \frac{2(x-iy)}{(x+iy)(x-iy)} + c$$

$$= (x^2 - y^2) + 2xiy - \frac{2(x-iy)}{x^2+y^2} + c$$

$$= \left(x^2 - y^2 - \frac{2x}{x^2+y^2}\right) + i\left(2xy + \frac{2y}{x^2+y^2}\right) + c$$

$$u + iv = \left[\frac{x^4 - y^4 - 2x}{x^2+y^2}\right] + i\left[\frac{2x^3y + 2xy^3 + 2y}{x^2+y^2}\right] + c$$

Equating the real and imaginary parts we observe that the real part u is same as in the given problem and the required imaginary part is given by,

$$v = \frac{2x^3y + 2xy^3 + 2y}{x^2+y^2}$$

---

14. *Find the analytic function* $f(z)$ *given* $u = e^{-x}\{(x^2-y^2)\cos y + 2xy \sin y\}$

>> $u = e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$

$$u_x = e^{-x}(2x\cos y + 2y \sin y) + \{(x^2 - y^2)\cos y + 2xy \sin y\}(-e^{-x})$$

$$u_y = e^{-x}\{(x^2-y^2)(-\sin y) + \cos y(-2y) + 2x(y\cos y + \sin y)\}$$

Consider $f'(z) = u_x + iv_x$. But $v_x = -u_y$ (*C-R equation*)

$\therefore \quad f'(z) = u_x - iu_y$

Putting $x = z,\ y = 0$, we have

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

i.e., $f'(z) = e^{-z}(2z) + z^2(-e^{-z}) - i\cdot 0 = (2z-z^2)e^{-z}$

$\therefore \quad f(z) = \int (2z - z^2)e^{-z}\,dz + c$

Integrating by applying Bernoulli's rule we have,

$$f(z) = (2z - z^2)(-e^{-z}) - (2-2z)(e^{-z}) + (-2)(-e^{-z}) + c$$

$$= -2z e^{-z} + z^2e^{-z} - 2e^{-z} + 2ze^{-z} + 2e^{-z} + c$$

Thus $f(z) = z^2e^{-z} + c$

---

15. *Determine the analytic function* $f(z) = u + iv$ *given that the real part* $u = e^{2x}(x \cos 2y - y \sin 2y)$

>> By data, $u = e^{2x}(x \cos 2y - y \sin 2y)$

$$u_x = e^{2x} \cdot \cos 2y + 2e^{2x}(x \cos 2y - y \sin 2y)$$

$$= e^{2x}(\cos 2y + 2x \cos 2y - 2y \sin 2y)$$

$$u_y = e^{2x}(-2x \sin 2y - 2y \cos 2y - \sin 2y)$$

$$= -e^{2x}(2x \sin 2y + 2y \cos 2y + \sin 2y)$$

Consider $f'(z) = u_x + i v_x = u_x - i u_y$ by $C-R$ equation.

Putting $x = z,\ y = 0$ we have,

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

ie., $f'(z) = e^{2z}(1 + 2z)$

$\therefore$ $f(z) = \int (1 + 2z)e^{2z}\, dz$

$$f(z) = (1 + 2z)\frac{e^{2z}}{2} - 2 \cdot \frac{e^{2z}}{4} = \frac{e^{2z}}{2} + z e^{2z} - \frac{e^{2z}}{2}$$

Thus $\boldsymbol{f(z) = z e^{2z} + c}$

Also $f(z) = u + iv = (x + iy)e^{2(x+iy)}$

$$= e^{2x}(x + iy)(\cos 2y + i \sin 2y)$$

$$f(z) = e^{2x}(x \cos 2y - y \sin 2y) + i e^{2x}(x \sin 2y + y \cos 2y)$$

---

16. *Find the analytic function* $f(z)$ *whose real part is* $\dfrac{\sin 2x}{\cos h\, 2y - \cos 2x}$ *and hence find the imaginary part.*

>> Let $u = \dfrac{\sin 2x}{\cos h\, 2y - \cos 2x}$

$$\therefore \quad u_x = \frac{(\cos h\, 2y - \cos 2x)(2 \cos 2x) - (\sin 2x)(2 \sin 2x)}{(\cos h\, 2y - \cos 2x)^2}$$

$$u_y = \frac{-\sin 2x\,(2 \sin h\, 2y)}{(\cos h\, 2y - \cos 2x)^2}$$

Consider $f'(z) = u_x + i v_x = u_x - i u_y$ by $C-R$ equation.

Putting $x = z,\ y = 0$ we have,

$$f'(z) = [u_x]_{(z,\,0)} - i[u_y]_{(z,\,0)}$$

ie.,
$$f'(z) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2} - i\cdot 0$$

$$f'(z) = \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2} = \frac{-2}{(1-\cos 2z)} = \frac{-2}{2\sin^2 z}$$

ie.,
$$f'(z) = -\operatorname{cosec}^2 z$$

Thus $f(z) = \cot z + c$

We shall separate $\cot z = \cot(x+iy)$ into real and imaginary parts to find v.
Consider $f(z) = \cot z$

ie.,
$$u + iv = \cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)}$$

ie.,
$$u + iv = \frac{\cos(x+iy)\sin(x-iy)}{\sin(x+iy)\sin(x-iy)}$$

$$= \frac{\frac{1}{2}\left[\sin(x-iy+x+iy) + \sin(x-iy-x-iy)\right]}{\frac{1}{2}\left[\cos(x+iy-x+iy) - \cos(x+iy+x-iy)\right]}$$

$$= \frac{\sin 2x + \sin(-2iy)}{\cos(2iy) - \cos 2x} = \frac{\sin 2x - i\sin h\,2y}{\cos h\,2y - \cos 2x}$$

$\therefore$
$$u + iv = \left[\frac{\sin 2x}{\cos h\,2y - \cos 2x}\right] + i\left[\frac{-\sin h\,2y}{\cos h\,2y - \cos 2x}\right]$$

(It may be observed that the real part u is the given problem)

Thus the required imaginary part $\boldsymbol{v = \dfrac{-\sin h\,2y}{\cos h\,2y - \cos 2x}}$

---

17. *If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = (x^2 - y^2) + \dfrac{x}{x^2 + y^2}$ find the complex potential as a function of the complex variable z and hence determine ϕ.*

>> $$\psi_x = 2x + \frac{(x^2 + y^2)1 - x \cdot 2x}{(x^2+y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\psi_y = -2y + \frac{(x^2+y^2)0 - x \cdot 2y}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

Consider $f'(z) = \phi_x + i\psi_x$ But $\phi_x = \psi_y$ (C-R *equation*)

i.e., $f'(z) = \psi_y + i\psi_x$

Putting $x = z,\ y = 0$ we have

$$f'(z) = [\psi_y]_{(z,\,0)} + i[\psi_x]_{(z,\,0)}$$

i.e., $$f'(z) = 0 + i\left(2z + \frac{-z^2}{(z^2)^2}\right) = i\left(2z - \frac{1}{z^2}\right)$$

$\therefore$ $$f(z) = i\int\left(2z - \frac{1}{z^2}\right)dz + c = i\left(z^2 + \frac{1}{z}\right) + c$$

Thus $$f(z) = i\left(z^2 + \frac{1}{z}\right) + c$$

To find ϕ we shall separate the RHS into real and imaginary parts.

i.e., $$\phi + i\psi = i\left\{(x+iy)^2 + \frac{1}{x+iy}\right\} + c$$

$$= i\left\{(x^2 + i^2y^2 + 2xiy) + \frac{x - iy}{(x+iy)(x-iy)}\right\} + c$$

$$= i\{(x^2 - y^2) + 2xiy\} + i\left\{\frac{x-iy}{x^2+y^2}\right\} + c$$

$$= i(x^2 - y^2) - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2} + c$$

$\therefore$ $$\phi + i\psi = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c$$

Equating the real and imaginary parts we observe that the imaginary part ψ is same as the given problem and the required real part is

$$\phi = -2xy + \frac{y}{x^2 + y^2}$$

---

18. *Construct the analytic function whose real part is* $r^2 \cos 2\theta$

>> By data $u = r^2 \cos 2\theta$

$$\therefore \quad u_r = 2r \cos 2\theta, \; u_\theta = -2r^2 \sin 2\theta$$

Consider $f'(z) = e^{-i\theta}(u_r + i v_r)$. But $v_r = \frac{-1}{r} u_\theta$ (C-R *equation*)

$$\therefore \quad f'(z) = e^{-i\theta}\left(2r\cos 2\theta + i \cdot \frac{-1}{r} \cdot -2r^2 \sin 2\theta\right)$$

$$= e^{-i\theta}(2r\cos 2\theta + i \cdot 2r \sin 2\theta)$$

$$= 2re^{-i\theta}(\cos 2\theta + i \sin 2\theta) \qquad \ldots(1)$$

Putting $r = z$ and $\theta = 0$ we have,

$$f'(z) = 2z \quad \text{and hence} \quad f(z) = \int 2z\, dz + c = z^2 + c$$

Thus $f(z) = z^2 + c$

Remark : *Referring to (1) we have,* $f'(z) = 2re^{-i\theta}(e^{2i\theta}) = 2re^{i\theta} = 2z$

$$\therefore \quad f(z) = z^2 + c$$

---

19. *Determine the analytic function* $f(z)$ *whose imaginary part is* $\left(r - \frac{k^2}{r}\right)\sin\theta$, $r \neq 0$. *Hence find the real part of* $f(z)$ *and prove that it is harmonic.*

>> Let $v = \left(r - \frac{k^2}{r}\right)\sin\theta$

$$\therefore \quad v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta, \; v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$$

Consider $f'(z) = e^{-i\theta}(u_r + i v_r)$. But $u_r = \frac{1}{r} v_\theta$ (*C-R equation*)

$$\therefore \quad f'(z) = e^{-i\theta}\left(\frac{1}{r} v_\theta + i v_r\right)$$

ie., $$f'(z) = e^{-i\theta}\left[\left(1-\frac{k^2}{r^2}\right)\cos\theta + i\left(1+\frac{k^2}{r^2}\right)\sin\theta\right] \quad \ldots (1)$$

$$= e^{-i\theta}\left[(\cos\theta + i\sin\theta) - \frac{k^2}{r^2}(\cos\theta - i\sin\theta)\right]$$

$$= e^{-i\theta}\left[e^{i\theta} - \frac{k^2}{r^2}e^{-i\theta}\right] = 1 - \frac{k^2}{(re^{i\theta})^2} = 1 - \frac{k^2}{z^2}$$

ie., $$f'(z) = 1 - \frac{k^2}{z^2}$$

$\therefore$ $$f(z) = \int\left(1-\frac{k^2}{z^2}\right)dz + c$$

Thus $$f(z) = \left(z + \frac{k^2}{z}\right) + c$$

Now, let us find $u(r, \theta)$ from $f(z)$ by putting $z = re^{i\theta}$

ie., $$u + iv = (re^{i\theta}) + \frac{k^2}{re^{i\theta}} = r(\cos\theta + i\sin\theta) + \frac{k^2}{r}(\cos\theta - i\sin\theta)$$

ie., $$u + iv = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$$

Thus the required real part $u = \left(r + \frac{k^2}{r}\right)\cos\theta$

Remark : *From (1) we can get $f'(z)$ as a function of z by putting $r = z$ and $\theta = 0$.*

Next we shall show that u is harmonic.

That is to show that $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \ldots (2)$

Consider $u = \left(r + \frac{k^2}{r}\right)\cos\theta$

$\therefore$ $$u_r = \left(1 - \frac{k^2}{r^2}\right)\cos\theta, \quad u_{rr} = \frac{2k^2}{r^3}\cos\theta$$

$$u_\theta = -\left(r + \frac{k^2}{r}\right)\sin\theta, \quad u_{\theta\theta} = -\left(r + \frac{k^2}{r}\right)\cos\theta$$

LHS of (2) now becomes

$$\frac{2k^2}{r^3}\cos\theta + \frac{1}{r}\cos\theta - \frac{k^2}{r^3}\cos\theta - \frac{1}{r}\cos\theta - \frac{k^2}{r^3}\cos\theta = 0$$

Thus $u = \left(r + \frac{k^2}{r}\right)\cos\theta$ **is harmonic.**

---

20. *Show that the following function u is harmonic. Also determine the corresponding analytic function $f(z)$.*

$$u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy.$$

>> $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

$$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$$

$$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2 \qquad \ldots (1)$$

$$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$$

$$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2 \qquad \ldots (2)$$

(1) + (2) gives $u_{xx} + u_{yy} = 0$. Thus u **is harmonic.**

Consider $f'(z) = u_x + i\,v_x$. But $v_x = -u_y$ (*C-R equation*)

i.e., $f'(z) = u_x - i\,u_y$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

i.e., $f'(z) = \cos z + 2z - i(2\cos z + 4z)$

Integrating w.r.t. z we get

$$f(z) = \sin z + z^2 - i(2\sin z + 2z^2) + c$$

i.e., $f(z) = (1-2i)z^2 + (1-2i)\sin z + c$

Thus $\boldsymbol{f(z) = (1-2i)(z^2 + \sin z) + c}$

Type-3 ***Finding the conjugate harmonic function and the analytic function.***

We have proved that the real and imaginary parts of an analytic function $f(z) = u + iv$ are harmonic. u and v are called conjugate harmonic functions. (*Harmonic Conjugates*) Given u we can find v and vice-versa.

Working procedure for problems

- Given u, we find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$
- We consider C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$
- Substituting for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ we obtain a system of two non homogenous PDE of the form $\frac{\partial v}{\partial y} = f(x, y)$; $\frac{\partial v}{\partial x} = g(x, y)$
- These can be solved by direct integration to obtain the required v.
- The same procedure is adopted to find u given v.
- Further $u + iv$ will give us $f(z)$ as a function of x, y.
  Putting $x = z$, $y = 0$ we can obtain $f(z)$ as a function of z.

---

21. *Show that* $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ *is harmonic and find its harmonic conjugate. Also find the corresponding analytic function* $f(z)$.

\>> $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$; $u_{xx} = 6x + 6$

$u_y = -6xy - 6y$; $u_{yy} = -6x - 6$

$\therefore$ $u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$ **Thus u is harmonic.**

Now consider C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$

Substituting for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ we have,

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x \quad ; \quad \frac{\partial v}{\partial x} = -(-6xy - 6y) = 6xy + 6y$$

$$\Rightarrow \quad v = \int (3x^2 - 3y^2 + 6x)\, dy + f(x) \quad ; \quad v = \int (6xy + 6y)\, dx + g(y)$$

$$\therefore \quad v = 3x^2 y - y^3 + 6xy + f(x) \quad ; \quad v = 3x^2 y + 6xy + g(y)$$

Now we have to properly choose $f(x)$ and $g(y)$ to obtain a unique expression for v.

Simple comparison yields $f(x) = 0$, $g(y) = -y^3$

(We look for functions of x only in the second expression of v and functions of y only in the first expression of v)

Thus $v = 3x^2 y - y^3 + 6xy$

The analytic function is $f(z) = u + iv$

i.e., $f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2 y - y^3 + 6xy)$

Putting $x = z,\ y = 0$ the required $f(z) = z^3 + 3z^2 + 1$.

---

22. *Determine which of the following function is harmonic. Find the conjugate harmonic function and express $u + iv$ as an analytic function of z.*

(*a*) $v = \log\sqrt{x+y}$ (*b*) $v = \cos x \sinh y$

>> $v = \log\sqrt{x+y} = \frac{1}{2}\log(x+y)$

$$v_x = \frac{1}{2}\cdot\frac{1}{x+y} \qquad v_{xx} = \frac{1}{2}\cdot\frac{-1}{(x+y)^2}$$

$$v_y = \frac{1}{2}\cdot\frac{1}{x+y} \qquad v_{yy} = \frac{1}{2}\cdot\frac{-1}{(x+y)^2}$$

$$\therefore \quad v_{xx} + v_{yy} = \frac{-1}{(x+y)^2} \neq 0.$$

Hence $v = \log\sqrt{x+y}$ is not harmonic.

Now consider $v = \cos x \sin h\, y$

$$v_x = -\sin x \sin h\, y \qquad v_{xx} = -\cos x \sin h\, y$$

$$v_y = \cos x \cos h\, y \qquad v_{yy} = \cos x \sin h\, y$$

$$v_{xx} + v_{yy} = 0$$

Thus $v = \cos x \sinh y$ **is harmonic.**

To find the harmonic conjugate, we consider *C-R* equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

Substituting for $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}$ we have

$$\frac{\partial u}{\partial x} = \cos x \cosh y \quad ; \quad \frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\Rightarrow \quad u = \int \cos x \cosh y\, dx + f(y) \quad ; \quad u = \int \sin x \sinh y\, dy + g(x)$$

$$\therefore \quad u = \sin x \cosh y + f(y) \quad ; \quad u = \sin x \cosh y + g(x)$$

We have to choose $f(y) = 0,\ g(x) = 0$ to get a unique expression for u.

Thus $u = \sin x \cosh y$

Also $f(z) = u + iv$

$$= \sin x \cosh y + i \cos x \sinh y.$$

Putting $x = z,\ y = 0$ we get

$$f(z) = \sin z, \quad \text{since } \cosh 0 = 1 = \cos 0,\ \sinh 0 = 0$$

Remark : $f(z) = u(x, y) + iv(x, y)$ *can be converted into a function of* $x + iy$

We have $f(z) = \sin x \cos h\, y + i \cos x \sin h\, y$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin(x + iy) = \sin z$$

Thus $f(z) = \sin z$

---

23. *Show that* $u = e^x(x\cos y - y\sin y)$ *is harmonic & find its harmonic conjugate. Also determine the corresponding analytic function.*

\>> $u = e^x(x\cos y - y\sin y)$

$u_x = e^x \cdot \cos y + (x\cos y - y\sin y)e^x$

i.e., $u_x = e^x(\cos y + x\cos y - y\sin y)$

Now $u_{xx} = e^x \cdot \cos y + (\cos y + x\cos y - y\sin y)e^x$

i.e., $u_{xx} = e^x(2\cos y + x\cos y - y\sin y)$ ... (1)

Also $u_y = e^x(-x\sin y - [y\cos y + \sin y])$

$= -e^x(x\sin y + y\cos y + \sin y)$

Now $u_{yy} = -e^x(x\cos y + [-y\sin y + \cos y] + \cos y)$

i.e., $u_{yy} = -e^x(2\cos y + x\cos y - y\sin y)$ ... (2)

(1) + (2) gives $u_{xx} + u_{yy} = 0$. Thus **u is harmonic.**

Now consider C-R equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = \dfrac{-\partial u}{\partial y}$

i.e., $\dfrac{\partial v}{\partial y} = e^x(\cos y + x\cos y - y\sin y)$ ... (3)

$$\frac{\partial v}{\partial x} = e^x \,(x \sin y + y \cos y + \sin y) \qquad \ldots (4)$$

From (3), $v = e^x \,[\,\int \cos y \; dy + x \int \cos y \; dy - \int y \sin y \, dy\,] + f(x)$

i.e., $\quad v = e^x \,[\sin y + x \sin y - (y \cdot - \cos y - 1 \cdot - \sin y)\,] + f(x)$

($\int y \,\sin y$ *is carried out by Bernoulli's rule*)

i.e., $\quad v = e^x \,[\sin y + x \sin y + y \cos y - \sin y\,] + f(x)$

$\therefore \quad v = x e^x \sin y + e^x y \cos y + f(x) \qquad \ldots (5)$

Also from (4) we have,

$$v = \sin y \int x e^x \, dx + y \cos y \int e^x dx + \sin y \int e^x dx + g(y)$$

i.e., $\quad v = \sin y \,(x e^x - e^x) + y \cos y \, e^x + \sin y \, e^x + g(y)$

$\therefore \quad v = x e^x \sin y + e^x y \cos y + g(y) \qquad \ldots (6)$

Comparing (5) and (6) we must choose $f(x) = 0, \; g(y) = 0$.

Thus the required $\mathbf{v = x e^x \sin y + e^x y \cos y = e^x (x \sin y + y \cos y)}$

Now $\quad f(z) = u + iv$

i.e., $\quad f(z) = e^x \,(x \cos y - y \sin y) + i e^x (x \sin y + y \cos y)$

Putting $x = z$ and $y = 0$ we get $f(z) = z e^z$

Remark : $f(z) = u(x, y) + i v(x, y)$ *can be converted into a function of* $x + iy$

$$f(z) = x e^x (\cos y + i \sin y) + y e^x \,(i \cos y - \sin y)$$
$$= x e^x \,(\cos y + i \sin y) + i y e^x \,(\cos y + i \sin y)$$
$$= e^x \,(\cos y + i \sin y)\,(x + iy)$$
$$= e^x \cdot e^{iy} \,(x + iy) = e^{x+iy} \,(x + iy) = e^z z$$

Thus $\quad f(z) = z e^z$

---

24. *If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$ find ϕ and also the complex potential as a function of z.*

>> $\quad \psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$

$$\psi_x = \frac{\partial \psi}{\partial x} = 2x + \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\psi_y = \frac{\partial \psi}{\partial y} = -2y + \frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Consider $C-R$ equations $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \ \frac{\partial \psi}{\partial x} = \frac{-\partial \phi}{\partial y}$

Substituting for $\frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ we have,

$$\frac{\partial \phi}{\partial x} = -2y - \frac{2xy}{(x^2+y^2)^2} \qquad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = -2x + \frac{x^2-y^2}{(x^2+y^2)^2} \qquad \ldots (2)$$

From (i) $\quad \phi = -2y \int dx - y \int \frac{2x\,dx}{(x^2+y^2)^2} + f(y)$

Put $\quad x^2 + y^2 = t \quad \therefore \quad 2x\,dx = dt$

Now, $\quad \phi = -2yx - y \int \frac{dt}{t^2} + f(y) = -2xy - y\left(\frac{-1}{t}\right) + f(y)$

i.e., $\quad \phi = -2xy + \frac{y}{x^2+y^2} + f(y) \qquad \ldots (3)$

Also from (2) $\quad \phi = -2x \int dy + \int \frac{x^2-y^2}{(x^2+y^2)^2}\,dy + g(x)$

i.e., $\quad \phi = -2xy + \int \frac{(x^2-y^2)\,dy}{(x^2+y^2)^2} + g(x)$

$$\phi = -2xy + \int \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}\,dy + g(x)$$

$$= -2xy + \int \frac{1}{x^2+y^2}\,dy - \int \frac{2y^2}{(x^2+y^2)^2}\,dy + g(x)$$

$$= -2xy + \frac{1}{x}\tan^{-1}\frac{y}{x} - \int \frac{2y^2\,dy}{(x^2+y^2)^2} + g(x)$$

Putting $y = x\tan\theta,\ dy = x\sec^2\theta\,d\theta$

$$= -2xy + \frac{1}{x}\tan^{-1}\frac{y}{x} - \frac{2}{x}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta + g(x)$$

$$= -2xy + \frac{1}{x}\tan^{-1}\frac{y}{x} - \frac{2}{x}\int \sin^2\theta\,d\theta + g(x)$$

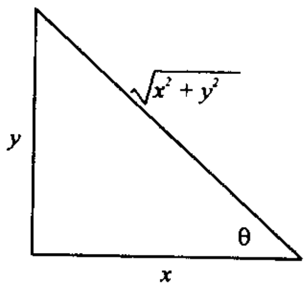
$$= -2xy + \frac{1}{x}\tan^{-1}\frac{y}{x} - \frac{1}{x}\int (1-\cos 2\theta)\,d\theta + g(x)$$

$$= -2xy + \frac{1}{x}\tan^{-1}\frac{y}{x} - \frac{1}{x}\left(\theta - \frac{\sin 2\theta}{2}\right) + g(x)$$

$$= -2xy + \frac{1}{x}\tan^{-1}\frac{y}{x} - \frac{1}{x}(\theta - \sin\theta\cos\theta) + g(x)$$

But $y = x\tan\theta \quad \therefore \quad \tan\theta = \frac{y}{x}$

From the figure



$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore \quad \phi = -2xy + \frac{1}{x}\tan^{-1}\frac{y}{x} - \frac{1}{x}\left(\tan^{-1}\frac{y}{x} - \frac{xy}{x^2+y^2}\right) + g(x)$$

i.e., $$\phi = -2xy + \frac{1}{x}\tan^{-1}\frac{y}{x} - \frac{1}{x}\tan^{-1}\frac{y}{x} + \frac{y}{x^2+y^2} + g(x)$$

i.e., $$\phi = -2xy + \frac{y}{x^2+y^2} + g(x) \qquad \dots(4)$$

Comparing (3) and (4) we must have $f(y) = 0$ and $g(x) = 0$

Thus the required $\phi = -2xy + \frac{y}{x^2+y^2}$

The complex potential is $f(z) = \phi + i\psi$

i.e., $$f(z) = \left[-2xy + \frac{y}{x^2+y^2}\right] + i\left[x^2 - y^2 + \frac{x}{x^2+y^2}\right]$$

Putting $x = z$ and $y = 0$ we get $f(z) = i\left(z^2 + \frac{1}{z}\right)$

Remark : *Compare this problem with Problem - 17 worked earlier wherein, starting from ψ we first obtained $f(z)$ by Milne Thompson method and later got ϕ by separating $f(z)$ into real and imaginary parts.*

It is important to note that if we have to construct $f(z)$ given u or v then Milne-Thompson method is easier (Type-2). However if we have to find $f(z)$ by obtaining v given u (or u given v) then the procedure as in Type-3 has to be employed.

25. *Given that $u = x^2 + 4x - y^2 + 2y$ as the real part of an analytic function, find v and hence find $f(z)$ in terms of z.*

>> $u = x^2 + 4x - y^2 + 2y$

$$\frac{\partial u}{\partial x} = 2x + 4 \ , \ \frac{\partial u}{\partial y} = -2y + 2$$

Consider $C - R$ equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

i.e., $\frac{\partial v}{\partial y} = 2x + 4 \quad ; \quad \frac{\partial v}{\partial x} = 2y - 2$

$\therefore \quad v = \int (2x + 4)\, dy + f(x) \ ; \ v = \int (2y - 2)\, dx + g(y)$

i.e., $v = 2xy + 4y + f(x) \quad ; \ v = 2xy - 2x + g(y)$

Comparing, we choose $f(x) = -2x,\ g(y) = 4y$

Thus the required $v = 2xy + 4y - 2x$

$\therefore \quad f(z) = u + iv = (x^2 + 4x - y^2 + 2y) + i(2xy + 4y - 2x)$

Putting $x = z$ and $y = 0$ we get $f(z) = z^2 + 4z - 2iz$

---

26. *Show that $u = \left(r + \frac{1}{r}\right) \cos\theta$ is harmonic. Find its harmonic conjugate and also the corresponding analytic function.*

>> $u = \left(r + \frac{1}{r}\right) \cos\theta$

We shall show that $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ ... (1)

$$u_r = \left(1 - \frac{1}{r^2}\right)\cos\theta \qquad u_{rr} = \frac{2}{r^3}\cos\theta$$

$$u_\theta = -\left(r + \frac{1}{r}\right)\sin\theta \qquad u_{\theta\theta} = -\left(r + \frac{1}{r}\right)\cos\theta$$

LHS of (1) now becomes,

$$\frac{2}{r^3}\cos\theta + \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta - \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta = 0$$

Thus the given u **is harmonic.**

To find v, let us consider $C - R$ equations in the polar form

$$r u_r = v_\theta \quad ; \quad r v_r = -u_\theta$$

ie., $v_\theta = \left(r - \frac{1}{r}\right)\cos\theta \quad ; \quad v_r = \left(1 + \frac{1}{r^2}\right)\sin\theta$

$\Rightarrow$ $v = \int\left(r - \frac{1}{r}\right)\cos\theta\, d\theta + f(r) \quad ; \quad v = \int\left(1 + \frac{1}{r^2}\right)\sin\theta\, dr + g(\theta)$

ie., $v = \left(r - \frac{1}{r}\right)\sin\theta + f(r) \quad ; \quad v = \left(r - \frac{1}{r}\right)\sin\theta + g(\theta)$

Comparing, we must have $f(r) = 0$ and $g(\theta) = 0$

Thus the required harmonic conjugate $v = \left(r - \frac{1}{r}\right)\sin\theta$

Also we have $f(z) = u + iv$

ie., $f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta} = re^{i\theta} + \frac{1}{re^{i\theta}} = z + \frac{1}{z}$$

Thus $f(z) = z + \frac{1}{z}$ is the analytic function.

Type-4 : *Miscellaneous problems*

27. *Find the analytic function* $f(z) = u + iv$ *given* $u - v = e^x(\cos y - \sin y)$

>> $u - v = e^x(\cos y - \sin y)$

We shall differentiate w.r.t. x and y partially.

$$u_x - v_x = e^x(\cos y - \sin y) \quad \ldots(1)$$

and $u_y - v_y = e^x(-\sin y - \cos y)$

Using C-R equations for the LHS of this equation in the form $u_y = -v_x$ and $v_y = u_x$ we have,

$$-v_x - u_x = e^x(-\sin y - \cos y)$$

or $u_x + v_x = e^x(\sin y + \cos y) \quad \ldots(2)$

Let us solve (1) and (2) simultaneously for u_x and v_x

$$(1)+(2): \; 2u_x = 2e^x \cos y \qquad \therefore \quad u_x = e^x \cos y$$

$$(1)-(2): \; -2v_x = -2e^x \sin y \qquad \therefore \quad v_x = e^x \sin y$$

We have $f'(z) = u_x + i v_x$

$$f'(z) = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y) = e^x \cdot e^{iy} = e^{x+iy} = e^z$$

Thus $f(z) = e^z + c$

---

28. *Find the analytic function $f(z)$ as a function of z given that the sum of its real and imaginary parts is $x^3 - y^3 + 3xy\,(x-y)$.*

>> Let $f(z) = u + iv$ be the analytic function and we have by data, $u + v = x^3 - y^3 + 3x^2 y - 3xy^2$

Differentiating w.r.t. x and y partially we have,

$$u_x + v_x = 3x^2 + 6xy - 3y^2 \qquad \ldots (1)$$

$$u_y + v_y = -3y^2 + 3x^2 - 6xy$$

Using C-R equations for the LHS of this equation in the form $u_y = -v_x$, and $v_y = u_x$ we have

$$-v_x + u_x = -3y^2 + 3x^2 - 6xy \qquad \ldots (2)$$

Now (1) + (2) gives, $2u_x = 6x^2 - 6y^2 \quad \therefore \; u_x = 3x^2 - 3y^2$

(1) − (2) gives, $2v_x = 12xy \quad \therefore \; v_x = 6xy$

Consider $f'(z) = u_x + i v_x = (3x^2 - 3y^2) + i(6xy)$

[**Note** : $f'(z) = 3(x^2 - y^2 + 2xiy) = 3(x+iy)^2 = 3z^2$]

Putting $x = z$ and $y = 0$ we get $f'(z) = 3z^2$

$\therefore \quad f(z) = \int 3z^2 \, dz + c$

Thus $f(z) = z^3 + c$

---

29. *If* $f(z) = u + iv$ *is analytic find* $f(z)$ *if* $u - v = (x-y)(x^2+4xy+y^2)$

>> $u - v = x^3 + 3x^2y - 3xy^2 - y^3$ on simplification.

$\therefore \quad u_x - v_x = 3x^2 + 6xy - 3y^2 \qquad \ldots (1)$

$u_y - v_y = 3x^2 - 6xy - 3y^2$

But $u_y = -v_x$ and $v_y = u_x$ by $C-R$ equations and hence we have,

$-v_x - u_x = 3x^2 - 6xy - 3y^2 \qquad \ldots (2)$

Let us solve for u_x and v_x from (1) and (2).

$(1)+(2): \quad -2v_x = 6(x^2 - y^2)$ or $v_x = 3(y^2 - x^2)$

$(1)-(2): \quad 2u_x = 12xy$ or $u_x = 6xy$

We have $f'(z) = u_x + iv_x$

ie., $f'(z) = 6xy + i.3(y^2 - x^2)$

Putting $x = z$ and $y = 0$ we get $f'(z) = -3iz^2$

$\therefore \quad f(z) = \int -3iz^2\,dz + c$

Thus $f(z) = -iz^3 + c$

---

30. *If* $f(z) = u(r, \theta) + iv(r, \theta)$ *is analytic and given that* $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$, $r \neq 0$ *determine the analytic function* $f(z)$.

>> $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$

Differentiating partially w.r.t r and also w.r.t θ we have,

$u_r + v_r = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta) \qquad \ldots (1)$

$u_\theta + v_\theta = \frac{-2}{r^2}(\sin 2\theta + \cos 2\theta)$

Using $C-R$ equations : $v_\theta = r\,u_r$ and $-u_\theta = r\,v_r$ we have

$-r\,v_r + r\,u_r = \frac{-2}{r^2}(\sin 2\theta + \cos 2\theta)$

or $$u_r - v_r = \frac{-2}{r^3}(\sin 2\theta + \cos 2\theta) \qquad \ldots (2)$$

Let us solve for u_r and v_r from (1) and (2).

$(1)+(2)$ gives $2u_r = \frac{-4}{r^3}\cos 2\theta$ or $u_r = \frac{-2}{r^3}\cos 2\theta$

$(1)-(2)$ gives $2v_r = \frac{4}{r^3}\sin 2\theta$ or $v_r = \frac{2}{r^3}\sin 2\theta$

We have $f'(z) = e^{-i\theta}(u_r + i v_r)$

$$f'(z) = \frac{-2e^{-i\theta}}{r^3}(\cos 2\theta - i\sin 2\theta) = \frac{-2}{r^3}e^{-3i\theta}$$

ie., $$f'(z) = \frac{-2}{(re^{i\theta})^3} = \frac{-2}{z^3} \quad \Rightarrow \quad f(z) = -2\int \frac{1}{z^3}\,dz + c$$

Thus $f(z) = \frac{1}{z^2} + c$

---

31. *If u and v are harmonic functions show that* $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ *is analytic.*

>> Let $P = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$, $Q = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

To prove that $P + iQ$ is analytic we shall show that C-R equations in the form $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ and $\frac{\partial Q}{\partial x} = \frac{-\partial P}{\partial y}$ are satisfied.

Consider $$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 v}{\partial y^2} = -\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0$$

since v is harmonic.

$$\therefore \quad \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0 \text{ or } \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$

Also consider $\dfrac{\partial P}{\partial y} + \dfrac{\partial Q}{\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right)$

$$= \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{since } u \text{ is harmonic.}$$

$$\therefore \quad \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0 \quad \text{or} \quad \frac{\partial Q}{\partial x} = \frac{-\partial P}{\partial y}$$

Thus $P + iQ$ is analytic.

Note : (*Alternative version of the problem*)

If $\quad P = \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}$ *and* $Q = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$,

show that $P + iQ$ *is analytic given that* $u + iv$ *is analytic.*

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32. *If* $f(z)$ *is analytic, show that* $\left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right] |f(z)|^2 = 4\,|f'(z)|^2$

>> Let $f(z) = u + iv$ be analytic.

$$\therefore \quad |f(z)| = \sqrt{u^2 + v^2} \quad \text{or} \quad |f(z)|^2 = u^2 + v^2 = \phi \ (say)$$

To prove that $\left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right] \phi = 4\,|f'(z)|^2$

That is to prove that $\phi_{xx} + \phi_{yy} = 4\,|f'(z)|^2$

Consider $\phi = u^2 + v^2$ and differentiate w.r.t. x partially.

$$\therefore \quad \phi_x = 2u\,u_x + 2v\,v_x = 2[u\,u_x + v\,v_x]$$

Differentiating w.r.t. x again we get

$$\phi_{xx} = 2[u\,u_{xx} + u_x^2 + v\,v_{xx} + v_x^2] \qquad \ldots(1)$$

Similarly we can also get

$$\phi_{yy} = 2[u\,u_{yy} + u_y^2 + v\,v_{yy} + v_y^2] \qquad \ldots(2)$$

Adding (1) and (2) we have,

$$\phi_{xx} + \phi_{yy} = 2[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2] \qquad \ldots(3)$$

Since $f(z) = u + iv$ is analytic, u and v are harmonic.

Hence $u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$. Further we also have C-R equations :
$v_y = u_x$, $u_y = -v_x$

Using these results in the RHS of (3), we have

$$\phi_{xx} + \phi_{yy} = 2[u \cdot 0 + v \cdot 0 + u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2]$$

i.e., $$\phi_{xx} + \phi_{yy} = 2[2u_x^2 + 2v_x^2] = 4[u_x^2 + v_x^2] \qquad \ldots (4)$$

But $f'(z) = u_x + i\,v_x$

$\therefore \quad |f'(z)| = \sqrt{u_x^2 + v_x^2}$ or $|f'(z)|^2 = u_x^2 + v_x^2$

Using this in the R.H.S of (4) we have $\boldsymbol{\phi_{xx} + \phi_{yy} = 4\,|f'(z)|^2}$

This proves the required result.

---

33. *If $f(z)$ is a regular function of z show that*

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

>> Let $f(z) = u + iv$ be the regular (*analytic*) function.

$$|f(z)| = \sqrt{u^2 + v^2} = \phi \text{ (say)}$$

We have to prove that $\left(\dfrac{\partial \phi}{\partial x}\right)^2 + \left(\dfrac{\partial \phi}{\partial y}\right)^2 = |f'(z)|^2$

That is to prove that $\phi_x^2 + \phi_y^2 = |f'(z)|^2$ where $\phi = \sqrt{u^2 + v^2}$

Consider $\phi^2 = u^2 + v^2$ (*squaring ϕ for convenience*)

Differentiating w.r.t. x partially we get,

$2\phi\,\phi_x = 2u\,u_x + 2v\,v_x$ and dividing by 2 we have,

$$\phi\,\phi_x = u\,u_x + v\,v_x \qquad \ldots (1)$$

Similarly we can also get

$$\phi\,\phi_y = u\,u_y + v\,v_y \qquad \ldots (2)$$

Squaring and adding (1) and (2) we have,

$$\phi^2(\phi_x^2 + \phi_y^2) = (u\,u_x + v\,v_x)^2 + (u\,u_y + v\,v_y)^2$$

$$= (u^2 u_x^2 + v^2 v_x^2 + 2\,uv\,u_x v_x) + (u^2 u_y^2 + v^2 v_y^2 + 2uv\,u_y v_y)$$

Since $f(z) = u + iv$ is analytic, we have C-R equations : $u_y = -v_x$ and $v_y = u_x$. By using these in the second bracket of the RHS we have,

$$\phi^2(\phi_x^2 + \phi_y^2) = (u^2 u_x^2 + v^2 v_x^2 + 2uv\,u_x v_x) + (u^2 v_x^2 + v^2 u_x^2 - 2uv\,u_x v_x)$$

$$= u^2(u_x^2 + v_x^2) + v^2(u_x^2 + v_x^2)$$

$$= (u_x^2 + v_x^2)(u^2 + v^2)$$

But $\phi^2 = u^2 + v^2$ and using this in the RHS we have,

$$\phi^2(\phi_x^2 + \phi_y^2) = \phi^2(u_x^2 + v_x^2)$$

or $$\phi_x^2 + \phi_y^2 = u_x^2 + v_x^2 \qquad \ldots (3)$$

But $f'(z) = u_x + i\,v_x$

$\therefore \ |f'(z)| = \sqrt{u_x^2 + v_x^2}$ or $|f'(z)|^2 = u_x^2 + v_x^2$

Using this in the RHS of (3) we get $\phi_x^2 + \phi_y^2 = |f'(z)|^2$

This proves the required result.

---

34. *If* $f(z)$ *is analytic show that* $\log|f(z)|$ *is harmonic.*

>> Let $f(z) = u + i\,v$ be analytic.

$$\therefore \quad \log|f(z)| = \log\sqrt{u^2 + v^2} = \frac{1}{2}\log(u^2 + v^2) = \phi \ (say)$$

We have to show that ϕ is harmonic. That is $\phi_{xx} + \phi_{yy} = 0$

Consider $\phi = \frac{1}{2}\log(u^2 + v^2)$ or $2\phi = \log_e(u^2 + v^2)$

or $\quad e^{2\phi} = u^2 + v^2$

Differentiating w.r.t. x partially we have,

$$e^{2\phi} \cdot 2\phi_x = 2\,u\,u_x + 2\,v\,v_x \text{ and dividing by 2 we get,}$$

$$e^{2\phi}\,\phi_x = u\,u_x + v\,v_x \qquad \ldots (1)$$

Differentiating again w.r.t. x by product rule we have,

$$e^{2\phi}\,\phi_{xx} + \phi_x\, e^{2\phi}\, 2\phi_x = u\,u_{xx} + u_x^2 + v\,v_{xx} + v_x^2$$

i.e., $$e^{2\phi}\,\phi_{xx} + 2\,e^{2\phi}\,\phi_x^2 = u\,u_{xx} + v\,v_{xx} + u_x^2 + v_x^2 \qquad \ldots (2)$$

Similarly we can also get

$$e^{2\phi}\,\phi_{yy} + 2e^{2\varphi}\,\phi_y^2 = u\,u_{yy} + v\,v_{yy} + u_y^2 + v_y^2 \qquad \ldots (3)$$

Adding (2) and (3) we have,

$$e^{2\phi}\,(\phi_{xx} + \phi_{yy}) + 2\,e^{2\phi}\,(\phi_x^2 + \phi_y^2)$$

$$= u\,(u_{xx} + u_{yy}) + v\,(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2 \qquad \ldots (4)$$

Since $f(z) = u + i\,v$ is analytic, u and v are harmonic.

$\therefore$ $$u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0.$$

Also we have *C-R* equations : $u_y = -v_x$ and $v_y = u_x$

Using these in the RHS of (4) we have

$$e^{2\phi}\,(\phi_{xx} + \phi_{yy}) + 2\,e^{2\phi}\,(\phi_x^2 + \phi_y^2) = u\cdot 0 + v\cdot 0 + u_x^2 + v_x^2 + v_x^2 + u_x^2$$

i.e., $$e^{2\phi}\,(\phi_{xx} + \phi_{yy}) + 2\,e^{2\phi}\,(\phi_x^2 + \phi_y^2) = 2\,(u_x^2 + v_x^2) \qquad \ldots (5)$$

But we have from (1) $e^{2\phi}\,\phi_x = u\,u_x + v\,v_x$

Now by squaring we have,

$$(e^{2\phi})^2\,\phi_x^2 = (u\,u_x + v\,v_x)^2 = u^2 u_x^2 + v^2 v_x^2 + 2\,u\,v\,u_x\,v_x$$

i.e., $$e^{4\phi}\,\phi_x^2 = u^2\,u_x^2 + v^2\,v_x^2 + 2\,u\,v\,u_x\,v_x \qquad \ldots (6)$$

Similarly we can also get,

$$e^{4\phi}\,\phi_y^2 = u^2 u_y^2 + v^2 v_y^2 + 2\,u\,v\,u_y\,v_y$$

But $u_y = -v_x$ and $v_y = u_x$ by *C–R* equations

$\therefore$ $$e^{4\phi}\,\phi_y^2 = u^2 v_x^2 + v^2 u_x^2 - 2\,u\,v\,(v_x)\,(u_x) \qquad \ldots (7)$$

Adding (6) and (7) we have

$$e^{4\phi}\,(\phi_x^2 + \phi_y^2) = u^2\,(u_x^2 + v_x^2) + v^2\,(u_x^2 + v_x^2)$$

i.e., $$e^{4\phi}\,(\phi_x^2 + \phi_y^2) = (u_x^2 + v_x^2)\,(u^2 + v^2)$$

Using $u^2 + v^2 = e^{2\phi}$ in the RHS we get,

$$e^{4\phi}(\phi_x^2 + \phi_y^2) = (u_x^2 + v_x^2)\, e^{2\phi}$$

Dividing by $e^{2\phi}$ we have, $e^{2\phi}(\phi_x^2 + \phi_y^2) = u_x^2 + v_x^2$

Using this result in the second term of the LHS of (5) we obtain,

$$e^{2\phi}(\phi_{xx} + \phi_{yy}) + 2(u_x^2 + v_x^2) = 2(u_x^2 + v_x^2)$$

i.e., $e^{2\phi}(\phi_{xx} + \phi_{yy}) = 0$. Dividing by $e^{2\phi}$ we have

$\phi_{xx} + \phi_{yy} = 0$. **Thus ϕ is harmonic.**

This proves the desired result.

---

35. *If $\phi(x, y)$ is a differentiable function and $f(z) = u(x, y) + i\,v(x, y)$ is a regular function, show that*

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = \left\{\left(\frac{\partial\phi}{\partial u}\right)^2 + \left(\frac{\partial\phi}{\partial v}\right)^2\right\} |f'(z)|^2$$

>> Let us treat ϕ as a composite function by regarding ϕ to be a function of u and v where u and v are functions of x, y.

By chain rule we have,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}\ ;\ \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}$$

ie., $\phi_x = \phi_u u_x + \phi_v v_x$ and $\phi_y = \phi_u u_y + \phi_v v_y$

$\therefore\quad \phi_x^2 + \phi_y^2 = (\phi_u u_x + \phi_v v_x)^2 + (-\phi_u v_x + \phi_v u_x)^2$, since $u_y = -v_x$ and $v_y = u_x$

Hence $\phi_x^2 + \phi_y^2 = \phi_u^2(u_x^2 + v_x^2) + \phi_v^2(u_x^2 + v_x^2)$

or $\quad \phi_x^2 + \phi_y^2 = (\phi_u^2 + \phi_v^2)(u_x^2 + v_x^2)$ ...(1)

But $\quad f'(z) = u_x + i\,v_x \quad \therefore\ |f'(z)|^2 = u_x^2 + v_x^2$

Now (1) becomes $\phi_x^2 + \phi_y^2 = (\phi_u^2 + \phi_v^2)\,|f'(z)|^2$

This proves the desired result.

3.6 Application to Flow Problems

We **recapitulate** some definitions and concepts already discussed in **Vector Calculus**.

Vector differential operator $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

If $\phi(x, y, z)$ is a scalar point function and $\vec{A} = A_1 i + A_2 j + A_3 k$ is a vector point function of x, y, z we have the following.

$$\nabla \phi = \text{grad}\, \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\nabla \cdot \vec{A} = \text{div}\, \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \vec{A} = \text{Curl}\, \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \text{Laplacian of } \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Geometrically, **grad** ϕ is a vector normal to the surface $\phi(x, y, z) = c$, c being a constant.

If $\vec{V}(x, y, z)$ represents any physical quantity, **div** $\vec{V}$ gives the rate at which the physical quantity is originating at that point per unit volume.

Supposing that a fluid is moving such that its velocity at any point is given by $\vec{V}(x, y, z)$ then **div** $\vec{V}$ gives the total gain in the volume of the fluid per unit volume per unit time.

div $\vec{V} = 0$ is the continuity equation of an incompressible fluid. Further a vector $\vec{A}$ whose divergence is zero is called a **solenoidal vector.**

Curl means rotation. A vector function $\vec{A}(x, y, z)$ is said to be **irrotational** if curl $\vec{A}$ is a null vector. Further if $\vec{A}(x, y, z)$ is irrotational, then there always exists a scalar function $\phi(x, y, z)$ such that $\nabla \phi = \vec{A}$. Then $\phi(x, y, z)$ is called the **scalar potential** of $\vec{A}$.

3.61 Velocity potential, Stream function, Equipotential lines, Stream lines and Complex potential

We discuss some application aspects based on the two properties of analytic functions.

Let us consider the irrotational motion of an incompressible fluid in two dimensions and suppose that the flow is in planes parallel to the x-y plane. Then the velocity $\vec{V}(x, y)$ of a fluid particle can be expressed as

$$\vec{V} = v_1 i + v_2 j \qquad \ldots (1)$$

Since the motion is irrotational there exists a scalar function $\phi(x, y)$ such that

$$\vec{V} = \nabla\phi(x, y) = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j \qquad \ldots (2)$$

Comparing (1) and (2) we have,

$$v_1 = \frac{\partial \phi}{\partial x} \text{ and } v_2 = \frac{\partial \phi}{\partial y} \qquad \ldots (3)$$

The scalar function $\phi(x, y)$ which gives the velocity components v_1 and v_2 is called the ***velocity potential.***

Further, since the fluid is incompressible we have $\text{div}\,\vec{V} = \nabla \cdot \vec{V} = 0$

That is, $\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right) \cdot \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j\right) = 0$

or $\quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \Rightarrow \phi(x, y)$ is a harmonic function.

That is to say that the velocity potential $\phi(x, y)$ is harmonic.

Hence $\phi(x, y)$ can be taken as the real part of an analytic function,

$$w = f(z) = \phi(x, y) + i\,\psi(x, y) \qquad \ldots (4)$$

$\psi(x, y)$ is the conjugate harmonic function & we give an interpretation for the same.

Consider $\psi(x, y) = c$ and differentiate w.r.t. x treating y as a function of x.

That is, $\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0$

or $$\frac{dy}{dx} = \frac{-\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = \frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}}$$ by using C - R equations.

That is, $\frac{dy}{dx} = \frac{v_2}{v_1}$ by using (3).

Since $\frac{dy}{dx}$ represents the slope at any point of the curve $\psi(x, y) = c$, we conclude that the resultant velocity $V = |\vec{V}| = \sqrt{v_1^2 + v_2^2}$ of the fluid particle is along the tangent to the curve $\psi(x, y) = c$. The fluid particles move along this curve.

$\psi(x, y)$ is called the ***stream function.***

The family of curves $\phi(x, y) =$ constant are known as ***equipotential lines*** and the family of curves $\psi(x, y) =$ constant are known as ***stream lines.***

Recalling the second property of analytic functions, [*Orthogonal property :Article* 3.52] we conclude that the equipotential lines $\phi(x, y) =$ constant and stream lines $\psi(x, y) =$ cosntant intersect each other orthogonally.

Also we have from (4),

$$\frac{dw}{dz} = f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \quad (\textit{Refer Theorem-1})$$

Using C-R equations we have,

$$\frac{dw}{dz} = f'(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

or $$\frac{dw}{dz} = f'(z) = v_1 - i\,v_2,$$ by using (3).

$$\therefore \quad \left|\frac{dw}{dz}\right| = |f'(z)| = \sqrt{v_1^2 + v_2^2} = |\vec{V}|, \text{ since } \vec{V} = v_1\,i + v_2\,j.$$

That is to say that the magnitude of fluid velocity is equal to the modulus of $f'(z)$.

The flow pattern fully represented by $w = f(z)$ is called the ***complex potential.***

Note : *In electrostatic problems* $\phi(x, y) = c_1$ *and* $\psi(x, y) = c_2$ *are respectively known as equipotential lines and lines of force.*

In heat flow problems $\phi(x, y) = c_1$ *and* $\psi(x, y) = c_2$ *are respectively known as isothermals and heat flow lines.*

WORKED PROBLEMS

36. *An electrostatic field in the x-y plane is given by the potential function* $\phi = 3x^2 y - y^3$. *Find the stream function.*

>> $\phi = 3x^2 y - y^3$, by data.

$$\therefore \quad \frac{\partial \phi}{\partial x} = 6xy \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2$$

But $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$ and $\dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$ by C - R equations.

Hence $\dfrac{\partial \psi}{\partial y} = 6xy$; $\dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \phi}{\partial y} = 3y^2 - 3x^2$

$$\Rightarrow \quad \psi = \int 6xy\,dy + f(x) \quad ; \quad \psi = \int (3y^2 - 3x^2)\,dx + g(y)$$

ie., $\psi = 3xy^2 + f(x)$; $\psi = 3xy^2 - x^3 + g(y)$

Let us choose $f(x) = -x^3$ and $g(y) = 0$, by comparison.

This will give us $\psi = 3xy^2 - x^3$

Thus the required stream function $\psi = 3xy^2 - y^3$

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37. *If the potential function is* log $(x^2 + y^2)$, *find the flux function and the complex potential.*

>> Let $\phi(x, y) = \log(x^2 + y^2)$

$$\therefore \quad \frac{\partial \phi}{\partial x} = \frac{2x}{x^2 + y^2} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \frac{2y}{x^2 + y^2}$$

But $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$ and $\dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$ by C-*R* equtions.

Hene $\dfrac{\partial \psi}{\partial y} = \dfrac{2x}{x^2 + y^2}$; $\dfrac{\partial \psi}{\partial x} = \dfrac{-2y}{x^2 + y^2}$

$$\Rightarrow \quad \psi = \int \frac{2x}{x^2 + y^2}\,dy + f(x) \quad ; \quad \psi = \int \frac{-2y}{x^2 + y^2}\,dx + g(y)$$

ie., $\psi = 2x \cdot \dfrac{1}{x} \tan^{-1} \dfrac{y}{x} + f(x)$; $\psi = -2y \cdot \dfrac{1}{y} \tan^{-1}\left(\dfrac{x}{y}\right) + g(y)$

or $\psi = 2\tan^{-1}\left(\dfrac{y}{x}\right) + f(x)$; $\psi = -2\tan^{-1}\left(\dfrac{x}{y}\right) + g(y)$

By choosing $f(x) = 0$ and $g(y) = 0$ we have,

$$\psi = 2\tan^{-1}(y/x) \quad \text{or} \quad \psi = -2\tan^{-1}(x/y)$$

Thus the required flux function is $\psi = 2\tan^{-1}(y/x)$

Further the complex potential is $w = f(z) = \phi + i\psi$

That is, $w = f(z) = \log(x^2+y^2) + i \cdot 2\tan^{-1}(y/x)$

By putting $x = z$ and $y = 0$ we obtain $w = f(z) = \log z^2 = 2\log z$

Thus the required complex potential $w = f(z) = 2\log z$

---

38. *Prove that $\phi = x^2 - y^2$ and $\psi = \dfrac{y}{x^2+y^2}$ are harmonic functions of (x, y) but are not harmonic conjugates*

>> Consider $\phi = x^2 - y^2$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = -2y, \quad \frac{\partial^2 \phi}{\partial x^2} = 2, \quad \frac{\partial^2 \phi}{\partial y^2} = -2$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 - 2 = 0 \Rightarrow \phi \textbf{ is harmonic.}$$

Next consider $\psi = \dfrac{y}{x^2+y^2}$

$$\frac{\partial \psi}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{(x^2+y^2)^2(-2y) + 2xy \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2y(x^2+y^2)\,[-(x^2+y^2)+4x^2]}{(x^2+y^2)^4}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3} \qquad \ldots (1)$$

Also, $$\frac{\partial \psi}{\partial y} = \frac{(x^2+y^2) \cdot 1 - y \cdot 2y}{(x^2+y^2)} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{(x^2+y^2)^2(-2y)-(x^2-y^2)\,2(x^2+y^2)\,2y}{(x^2+y^2)^4}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{2y(x^2+y^2)[-(x^2+y^2)-2(x^2-y^2)]}{(x^2+y^2)^4}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{2y(-3x^2+y^2)}{(x^2+y^2)^3} \qquad \ldots (2)$$

Adding (1) and (2) we obtain $\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2} = 0 \Rightarrow \psi$ **is harmonic.**

Let us find the harmonic conjugate of $\phi = x^2 - y^2$

We obtain $\frac{\partial \phi}{\partial x} = 2x$ and $\frac{\partial \phi}{\partial y} = -2y$

Since $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$ by C - R equations, we have,

$$\frac{\partial \psi}{\partial y} = 2x \quad ; \quad \frac{\partial \psi}{\partial x} = 2y$$

$$\Rightarrow \quad \psi = \int 2x\,dy + f(x) \quad ; \quad \psi = \int 2y\,dx + g(y)$$

ie., $\psi = 2xy + f(x) \quad ; \quad \psi = 2xy + g(y)$

By choosing $f(x) = 0 = g(y)$ we obtain $\psi = 2xy$.

The harmonic conjugate of $\phi = x^2 - y^2$ is $\psi = 2xy$ and $\psi = \frac{y}{x^2+y^2}$ by data.

Thus we conclude that ϕ and ψ are harmonic functions of (x, y) but are not harmonic conjugates.

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39. *In a two dimensional fluid flow if the velocity potential $\phi = e^{-x}\cos y + xy$, find the stream function.*

>> $\phi = e^{-x}\cos y + xy$, by data.

$$\therefore \quad \frac{\partial \phi}{\partial x} = -e^{-x}\cos y + y \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -e^{-x}\sin y + x$$

But $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$ by C - R equations.

Hence $\dfrac{\partial \psi}{\partial y} = -e^{-x}\cos y + y$, $\dfrac{\partial \varphi}{\partial x} = e^{-x}\sin y - x$

$\Rightarrow \quad \psi = \int(-e^{-x}\cos y + y)\,dy + f(x); \quad \psi = \int(e^{-x}\sin y - x)\,dx + g(y)$

$\therefore \quad \psi = -e^{-x}\sin y + \dfrac{y^2}{2} + f(x) \quad ; \quad \psi = -e^{-x}\sin y - \dfrac{x^2}{2} + g(y)$

By choosing $f(x) = -\dfrac{x^2}{2}$ and $g(y) = \dfrac{y^2}{2}$ we obtain $\psi = -e^{-x}\sin y + \dfrac{y^2}{2} - \dfrac{x^2}{2}$

Thus the required stream function $\psi = \dfrac{1}{2}(y^2 - x^2) - e^{-x}\sin y$

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40. *If $\Omega(z) = \log(z-a)$, $z \neq a$ represents the complex potential, show that the equipotential lines are a family of circles and stream lines are a family of straight lines.*

>> $\Omega(z) = \log(z-a)$

$\therefore \quad \phi + i\psi = \log(x + iy - a) = \log[(x-a) + iy]$

Using $\log(A + iB) = \log\sqrt{A^2 + B^2} + i\tan^{-1}(B/A)$ we have

$$\phi + i\psi = \log\sqrt{(x-a)^2 + y^2} + i\tan^{-1}\left[\frac{y}{x-a}\right]$$

$$\Rightarrow \quad \phi = \log\sqrt{(x-a)^2 + y^2} \text{ and } \psi = \tan^{-1}\left[\frac{y}{x-a}\right]$$

$\phi(x, y) = \text{constant}$ and $\psi(x, y) = \text{constant}$ respectively represents equipotential lines and stream lines.

That is, $\log\sqrt{(x-a)^2 + y^2} = c_1\ (say)$ and $\tan^{-1}\left[\dfrac{y}{x-a}\right] = c_2\ (say)$

or $\quad \sqrt{(x-a)^2 + y^2} = e^{c_1}$ and $\dfrac{y}{x-a} = \tan c_2$

or $\quad (x-a)^2 + y^2 = (e^{c_1})^2$ and $y = (\tan c_2)\,x - a\tan c_2$

or $\quad (x-a)^2 + y^2 = r^2 \quad \ldots(1)$

and $\quad y = mx + c \quad \ldots(2)$

where r, m, c are all arbitrary constants. It is evident that (1) is a circle with centre $(a, 0)$ and radius r and (2) is a straight line.

This proves the required result.

EXERCISES

Show that the following functions are analytic and hence find their derivative.

1. e^{2z} 2. $\cos z$ 3. $\sin hz$

4. $z^2 + 2z$ 5. $\sin 2z$

Construct the analytic function $f(z) = u + iv$ as a function z using the following data.

6. $u = e^x(x\cos y - y\sin y)$
7. $u = e^{-x}(x\cos y + y\sin y)$
8. $u = x\sin x\cos hy - y\cos x\sin hy$
9. $u = \log\sqrt{x^2 + y^2}$, $f(1) = 2i$
10. $v = e^{-x}(x\cos y + y\sin y)$
11. $v = \dfrac{y}{x^2 + y^2}$
12. $v = \left(r - \dfrac{1}{r}\right)\sin\theta$
13. $v = \dfrac{-\sin\theta}{r}$
14. $u = \dfrac{2\cos x\cos hy}{\cos 2x + \cos h2y}$
15. $u + v = (x+y) + e^x(\cos y + \sin y)$

Show that the following functions are harmonic and find the harmonic conjugates. Also find the corresponding analytic function **[16 to 20]**

16. $u = e^x\cos y + xy$
17. $u = (x-1)^3 - 3xy^2 + 3y^2$
18. $u = \dfrac{1}{r}\cos\theta$
19. $v = 2xy - 2x + 4y$
20. $v = e^{-2y}\sin 2x$
21. If $f(z) = u + iv$ is analytic and $v = 3x^2y - y^3$ find u. Also verify that $u = c_1$ and $v = c_2$, c_1 & c_2 are constants intersect each other orthogonally.
22. Show that $f(z) = \dfrac{x - iy}{x^2 + y^2}$ is holomorphic except at the origin.
23. Show that $f(z) = 2z + 3\bar{z}$ is not analytic.
24. Show that an analytic function with constant modulus is itself a constant.
25. Determine which of the following function is harmonic and find its harmonic conjugate. Also determine the corresponding analytic function.

    (a) $u = e^{2x}(x\cos 2y - y\sin 2y)$ (b) $u = e^{-2x}(x\cos 2y - y\sin 2y)$

ANSWERS

1. $2e^{2z}$
2. $-\sin z$
3. $\cosh z$
4. $2z+2$
5. $2\cos 2z$
6. ze^{z}
7. ze^{-z}
8. $z\sin z$
9. $\log z + 2i$
10. ize^{-z}
11. i/z
12. $z+(1/z)$
13. $1/z$
14. $\sec z$
15. $z + e^{z}$
16. $e^{x}\sin y - \frac{1}{2}(x^2-y^2)$; $e^{z} - i(z^2/2)$
17. $3x^2y - 6xy + 3y - y^3$; $(z-1)^3$
18. $\frac{-\sin\theta}{r}$; $\frac{1}{z}$
19. $x^2 - y^2 + 2(2x+y)$; $z^2 + (4-2i)z$
20. $e^{-2y}\cos 2x$; e^{2iz}
21. $u = x^3 - 3xy^2$
25. (a) $v = e^{2x}(x\sin 2y + y\cos 2y)$; ze^{2z}